A high order hybrid mimetic discretization on curvilinear quadrilateral meshes for complex geometries

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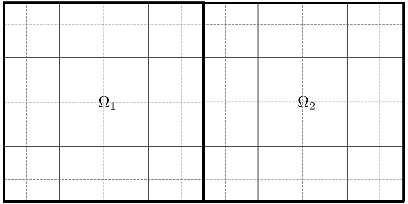
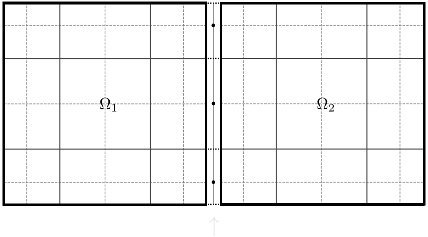
MS34B, 15 June 2018

Summary

Hybrid methods

Hybrid (finite element) methods are those methods that relax the continuity across the inter-element interface by introducing a Lagrange multiplier between elements.





For more information about hybrid methods, we refer to Pian [1], Raviart and Thomas [2], Brezzi and Fortin [3].

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2. Raviart, P.A. and Thomas, J.M. Primal hybrid finite element methods for 2nd order elliptic equations. Mathematics of computation, (1977) 31(138), 391-413.
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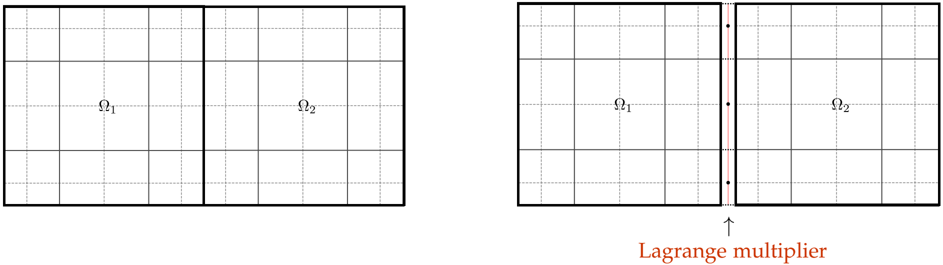

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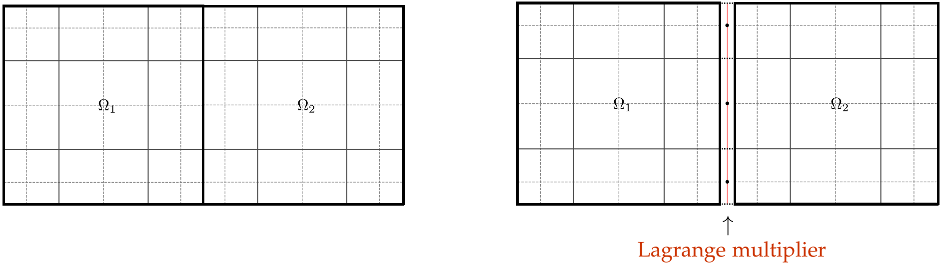

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Mimetic discretization

Mimetic methods aim to preserve the structure of partial differential equations at the discrete level.

A key feature of mimetic mixed finite element methods is that their function spaces satisfy the De Rham complex :

$$\begin{array}{ccccccccccc} \mathbb{R} & \rightarrow & \Omega^{(0)} & \xrightarrow{\text{grad}} & \Omega^{(1)} & \xrightarrow{\text{curl}} & \Omega^{(2)} & \xrightarrow{\text{div}} & \Omega^{(3)} & \rightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{R} & \rightarrow & \Omega_h^{(0)} & \xrightarrow{\text{grad}} & \Omega_h^{(1)} & \xrightarrow{\text{curl}} & \Omega_h^{(2)} & \xrightarrow{\text{div}} & \Omega_h^{(3)} & \rightarrow & 0. \end{array}$$

Therefore, mimetic methods are also called structure-preserving or compatible methods.

Hybrid Mimetic Spectral Element Method [4, 5, 6] is a high order mimetic mixed finite element method.

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Broken Sobolev spaces

Given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. A mesh, denoted by Ω_h, partitions Ω into K disjoint open elements Ω_k with Lipschitz boundary $\partial\Omega_k$,

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k,\ \Omega_i \cap \Omega_j = \emptyset,\ 1 \leq i \neq j \leq K.$$

We can break $H^1(\Omega), H(\mathrm{div},\Omega)$ and obtain the so-called broken Sobolev spaces[7] :

$$H^1(\Omega_h) = \left\{ \varphi \in L^2(\Omega) \,\middle|\, \varphi|_{\Omega_k} \in H^1(\Omega_k) \right\} = \prod_{k=1}^{K} H^1(\Omega_k),$$

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Spaces for interface functions are then defined as

$$H^{1/2}(\partial\Omega_h) := \mathrm{tr}^h_{\mathrm{grad}} H^1(\Omega), \qquad H^{-1/2}(\partial\Omega_h) := \mathrm{tr}^h_{\mathrm{div}} H(\mathrm{div},\Omega),$$

where trace operators $\mathrm{tr}^h_{\mathrm{grad}}$ and $\mathrm{tr}^h_{\mathrm{div}}$ restrict $\varphi \in H^1(\Omega)$ and $\boldsymbol{u} \in H(\mathrm{div},\Omega)$ respectively onto $\partial\Omega_h = \bigcup_{k=1}^{K} \partial\Omega_k$.

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Mixed formulation

We consider the constrained minimization problem,

$$\underset{\boldsymbol{u}\in L^2(\Omega)}{\arg}\ \min_{\operatorname{div}\boldsymbol{u}=-f} \frac{1}{2}\left(\boldsymbol{u},\boldsymbol{u}\right)_{L^2(\Omega)},$$

where f is given. By introducing a Lagrange multiplier φ, we can rewrite this constrained minimization problem into a saddle-point problem for $(\boldsymbol{u},\varphi)\in H(\operatorname{div},\Omega)\times H^1(\Omega)$:

$$\mathcal{L}(\boldsymbol{u},\varphi;f,\hat{\varphi}) = \frac{1}{2}\left(\boldsymbol{u},\boldsymbol{u}\right)_{L^2(\Omega)} + \left(\varphi,\operatorname{div}\,\boldsymbol{u}+f\right)_{L^2(\Omega)} - \left(\hat{\varphi},\operatorname{tr}_{\operatorname{div}}\boldsymbol{u}\right)_{L^2(\partial\Omega)},$$

where $\hat{\varphi}=\operatorname{tr}_{\operatorname{grad}}\varphi\in H^{1/2}(\partial\Omega)$ and $f\in L^2(\Omega)$ is given. Variational analysis on this functional gives rise to the mixed formulation : *Find* $(\boldsymbol{u},\varphi)\in H(\operatorname{div},\Omega)\times H^1(\Omega)$ *such that*

$$\begin{cases} \left(\boldsymbol{u},\boldsymbol{v}\right)_{L^2(\Omega)} + \left(\varphi,\operatorname{div}\,\boldsymbol{v}\right)_{L^2(\Omega)} & = \left(\hat{\varphi},\operatorname{tr}_{\operatorname{div}}\boldsymbol{v}\right)_{L^2(\partial\Omega)} \\ \left(\psi,\operatorname{div}\,\boldsymbol{u}\right)_{L^2(\Omega)} & = -\left(\psi,f\right)_{L^2(\Omega)} \end{cases},$$

for all $(\boldsymbol{v},\psi)\in H(\operatorname{div},\Omega)\times H^1(\Omega)$. This is a weak mixed formulation of the Poisson equation.

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Hybrid mixed formulation

If we set up a mesh Ω_h in Ω, by breaking $\boldsymbol{u}$ and φ into broken spaces, $H(\mathrm{div}, \Omega_h)$ and $H^1(\Omega_h)$, and introducing a new Lagrange multiplier $\check{\varphi}$ in the interface space $H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$, we can rewrite the functional as

$$\mathcal{L}(\boldsymbol{u}, \varphi, \check{\varphi}; f, \hat{\varphi}) = \frac{1}{2}\left(\boldsymbol{u}, \boldsymbol{u}\right)_{L^2(\Omega)} + \left(\varphi, \mathrm{div}\ \boldsymbol{u} + f\right)_{L^2(\Omega)} - \left(\check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u}\right)_{L^2(\partial\Omega_h \backslash \partial\Omega)} - \left(\hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u}\right)_{L^2(\partial\Omega)} .$$

The interface variable $\check{\varphi}$ serves as the Lagrange multiplier which enforces the continuity at the internal interface $\partial\Omega_h \backslash \partial\Omega$.

From this new functional, we can obtain the hybrid mixed formulation for the Poisson problem written as : *Given $f \in L^2(\Omega)$ and $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\ \varphi \in H^{1/2}(\partial\Omega)$, find $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$ such that*

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for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$. It is easy to prove that the interface variable $\check{\varphi}$ represents the restriction of φ onto $\partial\Omega_h \backslash \partial\Omega$.

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for all $(\boldsymbol{v},\psi,\check{\psi})\in H(\mathrm{div},\Omega_h)\times H^1(\Omega_h)\times H^{1/2}(\partial\Omega_h\backslash\partial\Omega)$. It is easy to prove that the interface variable $\check{\varphi}$ represents the restriction of φ onto $\partial\Omega_h\backslash\partial\Omega$.

Mimetic basis functions

Let $-1 = \xi_0 < \xi_1 < \cdots < \xi_N = 1$ be a partitioning of the interval $[-1, 1]$. The associated Lagrange polynomials are

$$h_i(\xi),\ \xi \in [-1, 1],\ i = 0, 1, \cdots, N,$$

which satisfy $h_i(\xi_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. The corresponding edge functions are then defined as,

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{\mathrm{d}h_k(\xi)}{\mathrm{d}\xi} = \sum_{k=i}^{N} \frac{\mathrm{d}h_k(\xi)}{\mathrm{d}\xi},\ i = 1, 2, \cdots, N,$$

which satisfy $\int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j}$.

Clearly, in $\mathbb{R}^2$, finite dimensional spaces spanned by basis functions $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\}$ and $\{e_i(\xi)e_j(\eta)\}$ satisfy the De Rham complex. Let vector-valued function $\boldsymbol{u}$ and scalar-valued function f be spanned into

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right) \quad \text{and} \quad f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \operatorname{div} \boldsymbol{u}$, then $f_h = \operatorname{div} \boldsymbol{u}_h$ and

$$f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta),$$

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which satisfy $\int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j}$.

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$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right) \quad \text{and} \quad f_h = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \operatorname{div} \boldsymbol{u}$, then $f_h = \operatorname{div} \boldsymbol{u}_h$ and

$$f_h = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta),$$

Mimetic basis functions

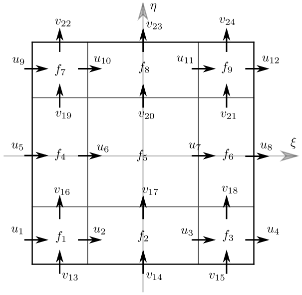

FIGURE – Reference domain.

Let vector-valued function $\boldsymbol{u}$ and scalar-valued function f be spanned into

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right), \ f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \mathrm{div}\ \boldsymbol{u}$, then

$$f_h = \mathrm{div}\ \boldsymbol{u}_h = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta).$$

Collect all equations and write them in vector form, we have

$$\underline{f} = \mathbb{E}^{2,1} \underline{u},$$

where

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

Mimetic basis functions

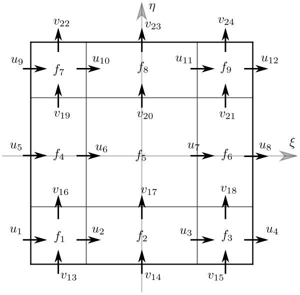

FIGURE – Reference domain.

Let vector-valued function $\boldsymbol{u}$ and scalar-valued function f be spanned into

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right), \; f_h = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \mathrm{div}\, \boldsymbol{u}$, then

$$f_h = \mathrm{div}\, \boldsymbol{u}_h = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta).$$

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where

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

Mimetic basis functions

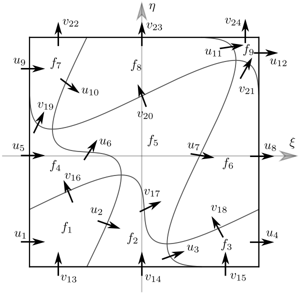

FIGURE – Curvilinear domain.

Let vector-valued function $\boldsymbol{u}$ and scalar-valued function f be spanned into

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right), \; f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \operatorname{div} \boldsymbol{u}$, then

$$f_h = \operatorname{div} \boldsymbol{u}_h = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta).$$

Collect all equations and write them in vector form, we have

$$\underline{f} = \mathbb{E}^{2,1} \underline{\boldsymbol{u}},$$

where

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

Mimetic trace basis functions

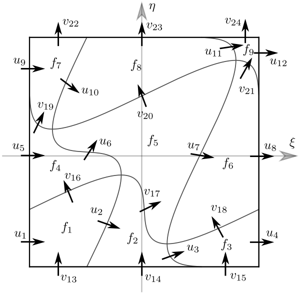

FIGURE – Curvilinear domain.

The trace variable $\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}$ can be discretized as

$$\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}_h = \left\{ \sum_{i=1}^{N} v_i^{\mathsf{s}} e_i(\xi),\ \sum_{i=1}^{N} v_i^{\mathsf{n}} e_i(\xi),\ \sum_{i=1}^{N} u_i^{\mathsf{w}} e_i(\eta),\ \sum_{i=1}^{N} u_i^{\mathsf{e}} e_i(\eta) \right\}.$$

There is a linear operator, $\mathbb{N}$, such that

$$\underline{\boldsymbol{u}}_{\mathrm{tr}} = \mathbb{N}\underline{\boldsymbol{u}},$$

where $\underline{\boldsymbol{u}}_{\mathrm{tr}} = \left(-\underline{v}_i^{\mathsf{s}}, \underline{v}_i^{\mathsf{n}}, -\underline{u}_i^{\mathsf{w}}, \underline{u}_i^{\mathsf{e}}\right)^T$ and

$$\mathbb{N} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

Mimetic trace basis functions

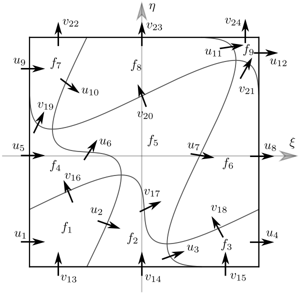

FIGURE – Curvilinear domain.

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There is a linear operator, $\mathbb{N}$, such that

$$\underline{\boldsymbol{u}}_{\mathrm{tr}} = \mathbb{N}\underline{\boldsymbol{u}},$$

where $\underline{\boldsymbol{u}}_{\mathrm{tr}} = \left(-\underline{v}_i^{\mathsf{s}}, \underline{v}_i^{\mathsf{n}}, -\underline{u}_i^{\mathsf{w}}, \underline{u}_i^{\mathsf{e}}\right)^T$ and

$$\mathbb{N} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

Dual representation

Let scalar functions p_h and q_h be expanded in terms of $\{e_i(\xi)e_j(\eta)\}$,

$$\left(p_h, q_h\right)_{L^2(\Omega_{\mathrm{ref}})} = \int_{\Omega_{\mathrm{ref}}} p_h(\xi,\eta) q_h(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta = \underline{\boldsymbol{p}}^T \mathbb{M}^{(2)} \underline{\boldsymbol{q}},$$

where $\mathbb{M}^{(2)}$ is the mass matrix. We define the dual basis functions[8] as

$$\left[\widetilde{e_1(\xi)e_1(\eta)}, \cdots, \widetilde{e_N(\xi)e_N(\eta)}\right] := \left[e_1(\xi)e_1(\eta), \cdots, e_N(\xi)e_N(\eta)\right] \mathbb{M}^{(2)^{-1}},$$

if we expand p_h in terms of the dual basis functions $\left\{\widetilde{e_i(\xi)e_j(\eta)}\right\}$, we have

$$\left(p_h, q_h\right)_{L^2(\Omega_{\mathrm{ref}})} = \underline{\tilde{\boldsymbol{p}}}^T \underline{\boldsymbol{q}},$$

where $\underline{\tilde{\boldsymbol{p}}} = \mathbb{M}^{(2)} \underline{\boldsymbol{p}}$. Furthermore, if $q_h = \mathrm{div}\ \boldsymbol{v}_h$, and $\boldsymbol{v}_h$ is expanded by basis functions $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\}$, we have

$$\left(\tilde{p}_h, \mathrm{div}\ \boldsymbol{v}_h\right)_{L^2(\Omega_{\mathrm{ref}})} = \underline{\tilde{\boldsymbol{p}}}^T \mathbb{E}^{2,1} \underline{\boldsymbol{v}}.$$

The same idea can be applied to trace basis functions.

8. Jain, V., Zhang, Y., Palha, A. and Gerritsma, M. Construction and application of algebraic dual polynomial representations for finite element methods. (2017) arXiv :1712.09472.

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$$(p_h, q_h)_{L^2(\Omega_{\mathrm{ref}})} = \int_{\Omega_{\mathrm{ref}}} p_h(\xi,\eta) q_h(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta = \underline{\boldsymbol{p}}^T \mathbb{M}^{(2)} \underline{\boldsymbol{q}},$$

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where $\mathbb{M}^{(2)}$ is the mass matrix. We define the dual basis functions[8] as

$$\left[\widetilde{e_1(\xi) e_1(\eta)}, \cdots, \widetilde{e_N(\xi) e_N(\eta)} \right] := \left[e_1(\xi) e_1(\eta), \cdots, e_N(\xi) e_N(\eta) \right] {\mathbb{M}^{(2)}}^{-1},$$

if we expand p_h in terms of the dual basis functions $\left\{ \widetilde{e_i(\xi) e_j(\eta)} \right\}$, we have

$$(p_h, q_h)_{L^2(\Omega_{\text{ref}})} = \underline{\tilde{\boldsymbol{p}}}^T \underline{\boldsymbol{q}},$$

where $\underline{\tilde{\boldsymbol{p}}} = \mathbb{M}^{(2)} \underline{\boldsymbol{p}}$. Furthermore, if $q_h = \operatorname{div} \boldsymbol{v}_h$, and $\boldsymbol{v}_h$ is expanded by basis functions $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\}$, we have

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Discretization

Hybrid mixed formulation

Given $f \in L^2(\Omega)$ *and* $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\ \varphi \in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$ *such that*

$$
\begin{cases}
(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)} + (\varphi, \mathrm{div}\ \boldsymbol{v})_{L^2(\Omega)} - (\check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v})_{L^2(\partial\Omega_h \backslash \partial\Omega)} & = (\hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v})_{L^2(\partial\Omega)} \\
(\psi, \mathrm{div}\ \boldsymbol{u})_{L^2(\Omega)} & = -(\psi, f)_{L^2(\Omega)} \\
-(\check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u})_{L^2(\partial\Omega_h \backslash \partial\Omega)} & = 0
\end{cases},
$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$.

We choose finite dimensional spaces spanned by following basis functions for the discretization :

- $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\} \rightarrow H(\mathrm{div}, \Omega_h)$.
- $\left\{\widetilde{e_i(\xi)e_j(\eta)}\right\} \rightarrow H^1(\Omega_h)$.
- $\left\{\widetilde{e_i(s)}\right\} \rightarrow H^{1/2}(\partial\Omega_h)$.
- $\{e_i(\xi)e_j(\eta)\} \rightarrow L^2(\Omega_h)$.

Discretization

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$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)} + (\varphi, \mathrm{div}\ \boldsymbol{v})_{L^2(\Omega)} - (\check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v})_{L^2(\partial\Omega_h \backslash \partial\Omega)} & = (\hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v})_{L^2(\partial\Omega)} \\ (\psi, \mathrm{div}\ \boldsymbol{u})_{L^2(\Omega)} & = -(\psi, f)_{L^2(\Omega)} \\ -(\check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u})_{L^2(\partial\Omega_h \backslash \partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$.

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- $\left\{e_i\widetilde{(\xi)e_j}(\eta)\right\} \to H^1(\Omega_h)$.
- $\left\{\widetilde{e_i(s)}\right\} \to H^{1/2}(\partial\Omega_h)$.
- $\{e_i(\xi)e_j(\eta)\} \to L^2(\Omega_h)$.

Discretization

Hybrid mixed formulation

Given $f \in L^2(\Omega)$ *and* $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\, \varphi \in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)} + (\varphi, \mathrm{div}\ \boldsymbol{v})_{L^2(\Omega)} - (\check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v})_{L^2(\partial\Omega_h \backslash \partial\Omega)} & = (\hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v})_{L^2(\partial\Omega)} \\ (\psi, \mathrm{div}\ \boldsymbol{u})_{L^2(\Omega)} & = -(\psi, f)_{L^2(\Omega)} \\ -(\check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u})_{L^2(\partial\Omega_h \backslash \partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h) \times H^{1/2}(\partial\Omega_h \backslash \partial\Omega)$.

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

- $\mathbb{M}^{(1)}$: metric-dependent ; element-wise-block-diagonal ;
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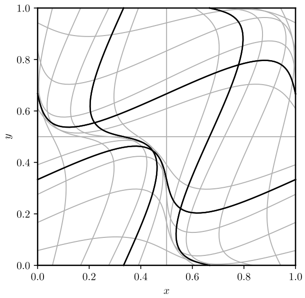

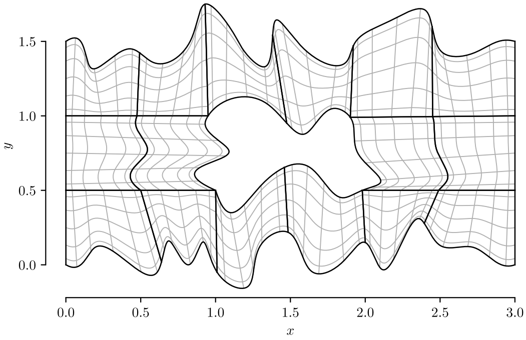

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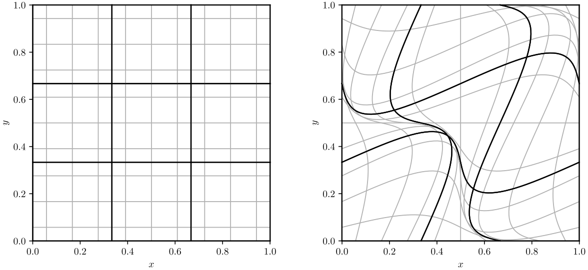
Manufactured solution

Given a domain $\Omega = [0,1]^2$ and an exact solution $\varphi_{\text{exact}} = \cos(3\pi x e^{y})$, we solve the Poisson problem with

$$
\begin{aligned}
f_{\text{exact}} &= -\text{div grad}\, \varphi_{\text{exact}} && \text{in } \Omega, \\
\hat{\varphi} &= \text{tr}_{\text{grad}}\, \varphi_{\text{exact}} && \text{on } \partial\Omega.
\end{aligned}
$$



$$
\begin{cases}
x = \dfrac{1}{2} + \dfrac{1}{2}\left[\xi + c\sin(\pi\xi)\sin(\pi\eta)\right] \\
y = \dfrac{1}{2} + \dfrac{1}{2}\left[\eta + c\sin(\pi\xi)\sin(\pi\eta)\right]
\end{cases}.
$$

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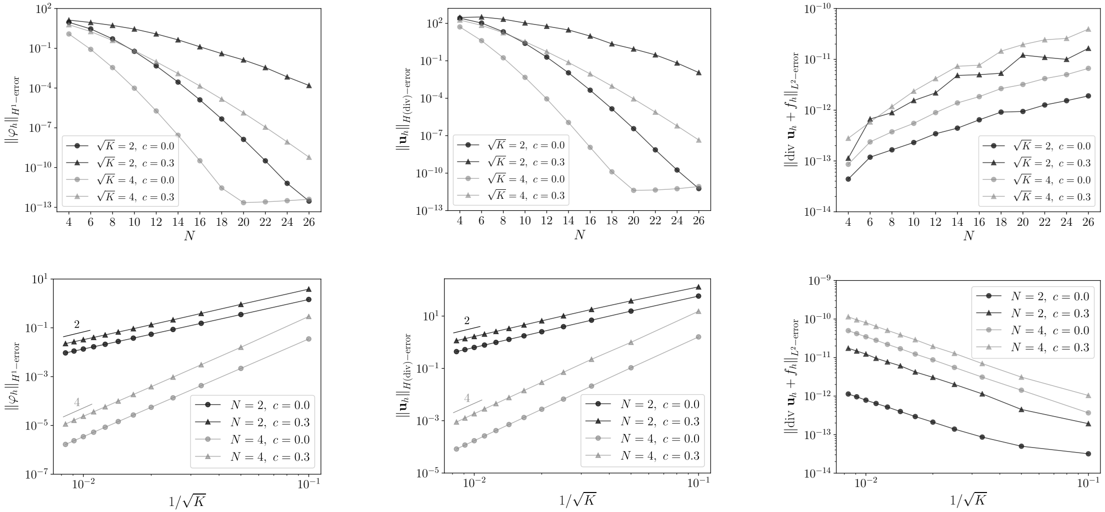
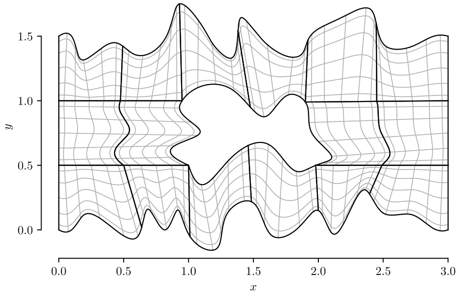
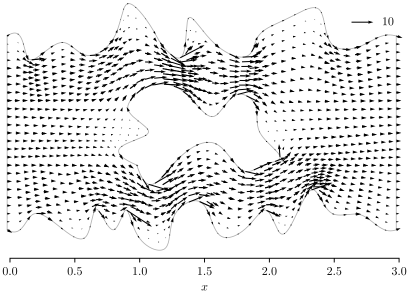


Potential flow in a domain with spline interpolation boundaries





- Upper, lower and inner boundaries : Cubic splines interpolated free-slip walls.
- Left and right boundaries : Inlet and outlet of potential difference $\Delta\varphi = 10$.

Boundary	Sequence of samples.
Lower	$(0,0)$, $(0.11,0.01)$, $(0.20,0.12)$, $(0.61,-0.05)$, $(0.69,0.16)$, $(0.82,0)$, $(0.91,0.15)$, $(1.01,-0.05)$, $(1.21,-0.15)$, $(1.30,0.13)$, $(1.48,0.22)$, $(1.65,-0.05)$, $(1.85,0.02)$, $(2,0.15)$, $(2.11,-0.03)$, $(2.36,0.31)$, $(2.50,0.13)$, $(2.71,0.12)$, $(2.91,0)$, $(3,0)$.
Upper	$(0,1.5)$, $(0.09,1.51)$, $(0.17,1.32)$, $(0.43,1.45)$, $(0.58,1.36)$, $(0.83,1.50)$, $(0.93,1.75)$, $(1.14,1.52)$, $(1.18,1.45)$, $(1.33,1.33)$, $(1.4,1.64)$, $(1.59,1.45)$, $(1.88,1.37)$, $(1.92,1.47)$, $(2.15,1.63)$, $(2.40,1.71)$, $(2.51,1.43)$, $(2.72,1.42)$, $(2.89,1.5)$, $(3,1.5)$.
Inner	$(1,0.5)$, $(1.11,0.35)$, $(1.32,0.55)$, $(1.62,0.66)$, $(1.85,0.45)$, $(1.98,0.5)$, $(2.1,0.55)$, $(1.95,0.75)$, $(1.9,0.99)$, $(1.79,1.05)$, $(1.6,0.88)$, $(1.33,1.09)$, $(0.95,1)$, $(0.93,0.95)$, $(1.09,0.76)$, $(0.89,0.65)$, $(1,0.5)$.

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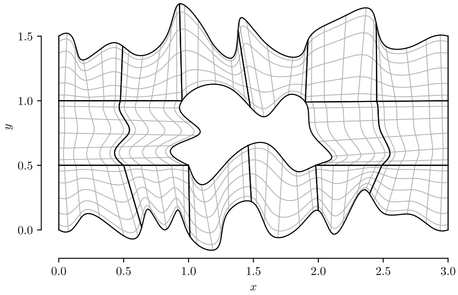

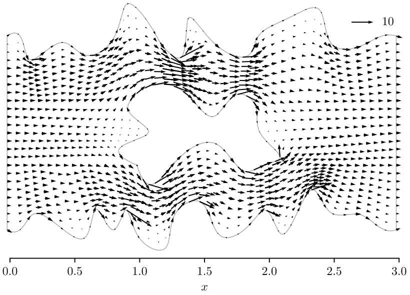

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TABLE – Fluxes through the domain.

N	Number of elements				
	16	64	256	576	1024
2	2.49949	2.92468	2.95905	3.01901	3.02207
4	2.95266	3.03115	3.02979	3.03123	3.03129
6	3.04810	3.02942	3.03120	3.03139	3.03139
8	3.01246	3.03047	3.03137	3.03140	3.03141
10	3.02062	3.03108	3.03141	3.03141	3.03141
12	3.03175	3.03137	3.03141	3.03141	3.03141
14	3.03045	3.03142	3.03141	3.03141	3.03141

Conclusions

We have proposed a high order spectral element method using dual polynomials :

- The method is hybrid ; it is very much parallelizable.
- The method is mimetic ; the divergence operator is preserved at the discrete level.
- Only one block is metric-dependent. Remaining blocks are metric-free, extremely sparse and finite-difference(volume)-like (containing non-zero entries of -1 and 1 only).
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