Kronecker delta

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1 Node polynomials

The Kronecker delta property for the basis node polynomials is obvious,

$$\mathrm{lll}_{\Phi}^{i,j,k}(P_{l,m,n}^{\Phi}) = \mathrm{lll}^{i,j,k}\left(\Phi^{-1}(P_{l,m,n}^{\Phi})\right) = \mathrm{lll}^{i,j,k}(P_{l,m,n}) = \delta_{l,m,n}^{i,j,k}\,.$$

□

2 Edge polynomials

The Kronecker delta properties for the basis edge polynomials are expressed as

$$\int_{E_{l,m,n}^{\Phi,\xi}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = \delta_{l,m,n}^{i,j,k},\quad \int_{E_{l,m,n}^{\Phi,\eta}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0,\quad \int_{E_{l,m,n}^{\Phi,\varsigma}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0, \tag{1a}$$

$$\int_{E_{l,m,n}^{\Phi,\xi}} \mathbf{lel}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0,\quad \int_{E_{l,m,n}^{\Phi,\eta}} \mathbf{lel}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = \delta_{l,m,n}^{i,j,k},\quad \int_{E_{l,m,n}^{\Phi,\varsigma}} \mathbf{lel}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0, \tag{1b}$$

$$\int_{E_{l,m,n}^{\Phi,\xi}} \mathbf{lle}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0,\quad \int_{E_{l,m,n}^{\Phi,\eta}} \mathbf{lle}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0,\quad \int_{E_{l,m,n}^{\Phi,\varsigma}} \mathbf{lle}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = \delta_{l,m,n}^{i,j,k}. \tag{1c}$$

We take the first equation in (1a) as an example.

$$\begin{aligned}
\int_{E_{l,m,n}^{\Phi,\xi}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} &= \int_{E_{l,m,n}^{\Phi,\xi}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\boldsymbol{x}_{\xi}\ \mathrm{d}\xi \\
&= \int_{E_{l,m,n}^{\xi}} \left(\left(\mathcal{J}^{-1}\right)^{\mathsf{T}}\begin{bmatrix}\mathrm{ell}^{i,j,k}\left(\xi,\eta,\varsigma\right)\\ 0\\ 0\end{bmatrix}\right)\cdot\begin{bmatrix}\mathcal{J}_{1,1}\\ \mathcal{J}_{2,1}\\ \mathcal{J}_{3,1}\end{bmatrix}\mathrm{d}\xi \\
&= \int_{E_{l,m,n}^{\xi}} \mathrm{ell}^{i,j,k}\left(\xi,\eta,\varsigma\right)\mathrm{d}\xi \\
&= \delta_{l,m,n}^{i,j,k},
\end{aligned} \tag{2}$$

where we have used the fact that $\mathcal{J}^{-1}\mathcal{J} = \mathcal{I}$. Meanwhile, we can find

$$\mathbf{ell}_{\Phi}^{i,j,k}\cdot\boldsymbol{x}_{\eta} = 0 \quad\text{and}\quad \mathbf{ell}_{\Phi}^{i,j,k}\cdot\boldsymbol{x}_{\varsigma} = 0.$$

Therefore, we can prove

$$\int_{E_{l,m,n}^{\Phi,\eta}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0 \quad\text{and}\quad \int_{E_{l,m,n}^{\Phi,\varsigma}} \mathbf{ell}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{r} = 0,$$

Analogously, we can prove (1b) and (1c). □

3 Face polynomials

The Kronecker delta properties for the basis face polynomials are expressed as

$$\int_{F_{l,m,n}^{\Phi,\xi}} \mathbf{lee}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = \delta_{l,m,n}^{i,j,k},\quad \int_{F_{l,m,n}^{\Phi,\eta}} \mathbf{lee}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = 0,\quad \int_{F_{l,m,n}^{\Phi,\varsigma}} \mathbf{lee}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = 0, \tag{3a}$$

$$\int_{F_{l,m,n}^{\Phi,\xi}} \mathbf{ele}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = 0,\quad \int_{F_{l,m,n}^{\Phi,\eta}} \mathbf{ele}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = \delta_{l,m,n}^{i,j,k},\quad \int_{F_{l,m,n}^{\Phi,\varsigma}} \mathbf{ele}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = 0, \tag{3b}$$

$$\int_{F_{l,m,n}^{\Phi,\xi}} \mathbf{eel}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = 0,\quad \int_{F_{l,m,n}^{\Phi,\eta}} \mathbf{eel}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = 0,\quad \int_{F_{l,m,n}^{\Phi,\varsigma}} \mathbf{eel}_{\Phi}^{i,j,k}\cdot\mathrm{d}\boldsymbol{A} = \delta_{l,m,n}^{i,j,k}. \tag{3c}$$

We take the first equation of (3a) as an example.

$$
\begin{aligned}
\int_{F^{\Phi,\xi}_{l,m,n}} \mathbf{lee}^{i,j,k}_{\Phi} \cdot \mathrm{d}\boldsymbol{A} &= \int_{F^{\xi}_{l,m,n}} \mathbf{lee}^{i,j,k}_{\Phi} \cdot (\boldsymbol{x}_\eta \times \boldsymbol{x}_\varsigma)\, \mathrm{d}\eta\mathrm{d}\varsigma \\
&= \int_{F^{\xi}_{l,m,n}} \left(\frac{\mathcal{J}}{\sqrt{g}} \begin{bmatrix} \mathrm{lee}^{i,j,k}(\xi,\eta,\varsigma) \\ 0 \\ 0 \end{bmatrix} \right) \cdot \sqrt{g} \begin{bmatrix} \mathcal{J}^{-1}_{1,1} \\ \mathcal{J}^{-1}_{1,2} \\ \mathcal{J}^{-1}_{1,3} \end{bmatrix} \mathrm{d}\eta\mathrm{d}\varsigma \\
&= \int_{F^{\xi}_{l,m,n}} \mathrm{lee}^{i,j,k}(\xi,\eta,\varsigma)\, \mathrm{d}\eta\mathrm{d}\varsigma \\
&= \delta^{i,j,k}_{l,m,n},
\end{aligned} \tag{4}
$$

where we have used

$$
\boldsymbol{x}_\eta \times \boldsymbol{x}_\varsigma = \begin{bmatrix} \mathcal{J}_{1,2} \\ \mathcal{J}_{2,2} \\ \mathcal{J}_{3,2} \end{bmatrix} \times \begin{bmatrix} \mathcal{J}_{1,3} \\ \mathcal{J}_{2,3} \\ \mathcal{J}_{3,3} \end{bmatrix} = \sqrt{g} \begin{bmatrix} \mathcal{J}^{-1}_{1,1} \\ \mathcal{J}^{-1}_{1,2} \\ \mathcal{J}^{-1}_{1,3} \end{bmatrix}
$$

and the fact $\mathcal{J}^{-1}\mathcal{J} = \mathcal{I}$. We can also find that

$$
\mathbf{lee}^{i,j,k}_{\Phi} \cdot (\boldsymbol{x}_\varsigma \times \boldsymbol{x}_\xi) = 0 \quad \text{and} \quad \mathbf{lee}^{i,j,k}_{\Phi} \cdot (\boldsymbol{x}_\xi \times \boldsymbol{x}_\eta) = 0.
$$

Therefore, we can prove the other two equations in (3a), and, similarly, (3b) and (3c). □

4 Volume polynomials

The Kronecker delta property for the basis volume polynomials is written as

$$
\int_{V^{\Phi}_{l,m,n}} \mathrm{eee}^{i,j,k}_{\Phi}(x,y,z)\mathrm{d}V = \delta^{i,j,k}_{l,m,n}.
$$

It can be proven through

$$
\begin{aligned}
\int_{V^{\Phi}_{l,m,n}} \mathrm{eee}^{i,j,k}_{\Phi}(x,y,z)\mathrm{d}V &= \int_{V^{\Phi}_{l,m,n}} \frac{1}{\sqrt{g}} \mathrm{eee}^{i,j,k}\left(\Phi^{-1}(x,y,z)\right) \mathrm{d}V \\
&= \int_{V_{l,m,n}} \mathrm{eee}^{i,j,k}(\xi,\eta,\varsigma)\mathrm{d}V \\
&= \delta^{i,j,k}_{l,m,n}.
\end{aligned}
$$

□