Hybrid Dual Finite Element Methods

Third Workshop on Minimum Residual & Least-Squares Finite Element Methods

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2nd October 2017

Differential forms

Differential forms

- wedge product : (**metric free**)

$$\wedge : \Lambda^k(\Omega) \times \Lambda^l(\Omega) \rightarrow \Lambda^{k+l}(\Omega)$$

- exterior derivative : (**metric free**)

$$\mathrm{d} : \Lambda^k(\Omega) \rightarrow \Lambda^{k+1}(\Omega)$$

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- ⋇ Leibniz rule : $\mathrm{d}\left(a^{(k)} \wedge b^{(l)}\right) = \mathrm{d}a^{(k)} \wedge b^{(l)} + (-1)^k a^{(k)} \wedge \mathrm{d}b^{(k)}$

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- **Hodge $\star$:** $\Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$; $\quad \Lambda^k(\Omega),\ \Lambda^{n-k}(\Omega)$ differently oriented
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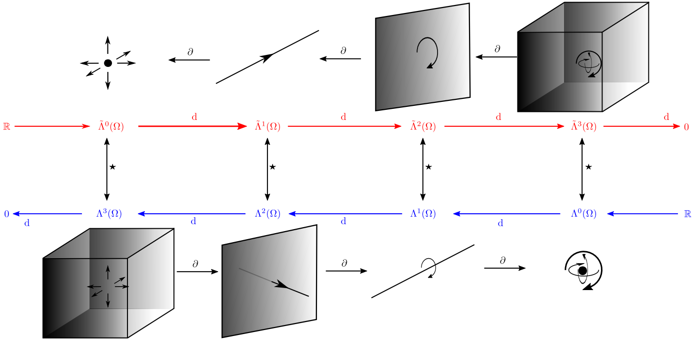
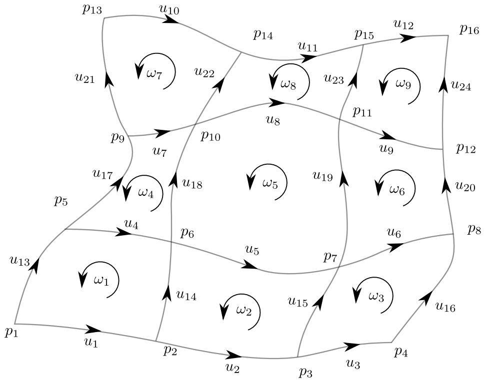


Algebraic topology



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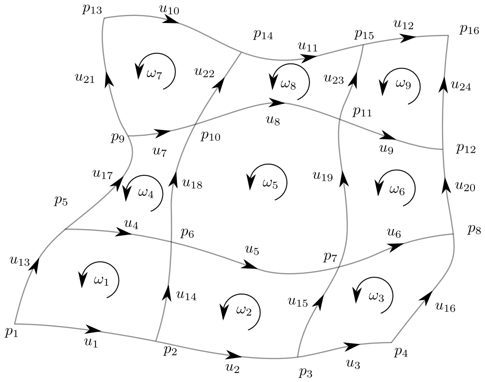
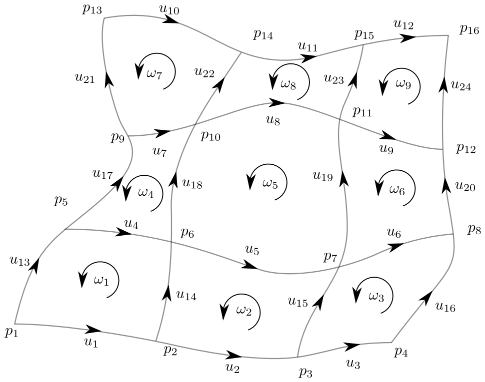


$$u^{(1)} = \mathrm{d}p^{(0)}$$

$$\begin{pmatrix} u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\\u_{13}\\u_{14}\\u_{15}\\u_{16}\\u_{17}\\u_{18}\\u_{19}\\u_{20}\\u_{21}\\u_{22}\\u_{23}\\u_{24} \end{pmatrix} = \begin{pmatrix}
-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1\\
-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1
\end{pmatrix} \begin{pmatrix} p_1\\p_2\\p_3\\p_4\\p_5\\p_6\\p_7\\p_8\\p_9\\p_{10}\\p_{11}\\p_{12}\\p_{13}\\p_{14}\\p_{15}\\p_{16} \end{pmatrix}$$

$$\boldsymbol{u} = \mathbb{E}^{1,0}\boldsymbol{p}$$

Algebraic topology

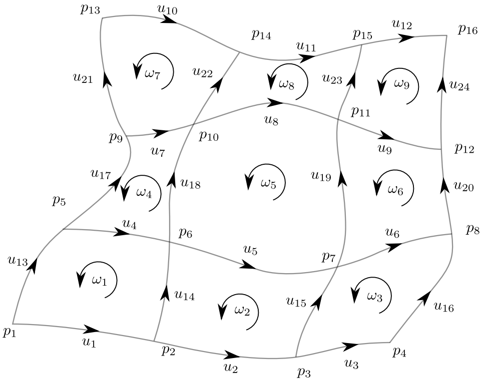


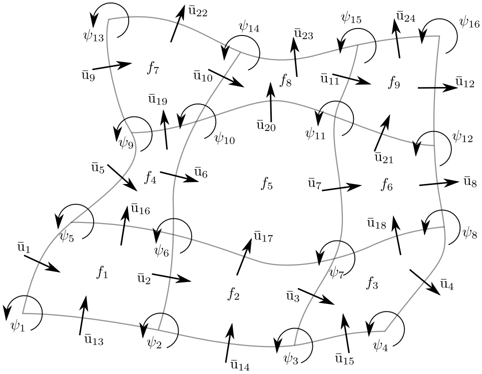
$$\omega^{(2)} = \mathrm{d}u^{(1)}$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \\ \omega_7 \\ \omega_8 \\ \omega_9 \end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \cdots \\ u_{24} \end{pmatrix}$$

$$\boldsymbol{\omega} = \mathbb{E}^{2,1}\boldsymbol{u}$$

Algebraic topology





Basis functions

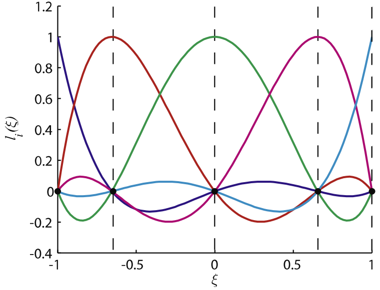


FIGURE – Lagrange polynomials $(N = 4)$ on Gauss–Lobatto mesh

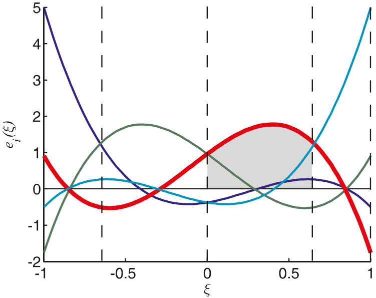


FIGURE – Edge polynomials $(N = 4)$ on Gauss–Lobatto mesh

Toward hybridization

Dual finite element method

⇓

Hybrid dual finite element method

Petrov–Galerkin method

⇓

Discontinuous Petrov–Galerkin method

Hybridization

Hybridization

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2nd order elliptic problem

We consider a constrainted minimization problem given as

$$\min_{\mathrm{d}u^{(n-1)}=-f^{(n)}} \frac{1}{2}\left(u^{(n-1)},u^{(n-1)}\right)_\Omega$$

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$$\mathcal{L}(\tilde{\phi}^{(0)}, u^{(n-1)}) = \frac{1}{2}\left(u^{(n-1)}, u^{(n-1)}\right)_\Omega - \int_\Omega \left(\mathrm{d}u^{(n-1)} + f^{(n)}\right) \wedge \tilde{\phi}^{(0)},$$

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$$\begin{cases} u^{(n-1)}=\star\mathrm{d}\tilde{\phi}^{(0)} & \text{in } \Omega \\ -\mathrm{d}u^{(n-1)}=f^{(n)} & \text{in } \Omega \\ \mathrm{tr}\,\tilde{\phi}^{(0)}=0 & \text{on } \Gamma \end{cases}$$

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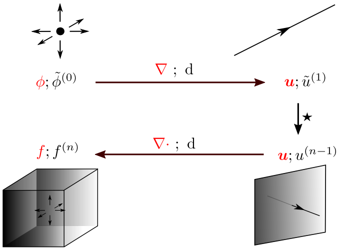

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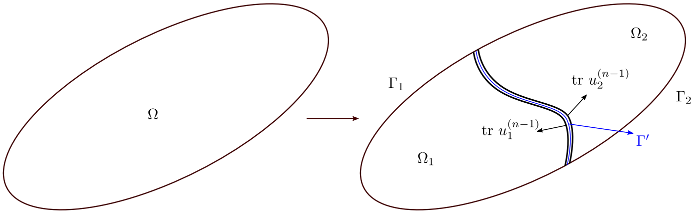

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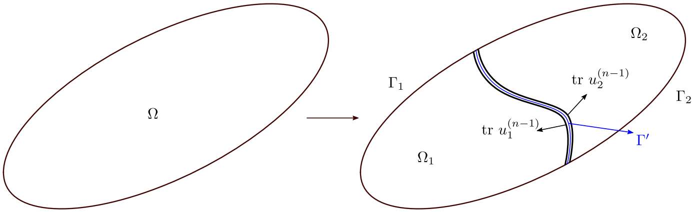

$$\begin{cases} u^{(n-1)} = \star \mathrm{d}\tilde{\phi}^{(0)} & \text{in } \Omega \\ -\mathrm{d}u^{(n-1)} = f^{(n)} & \text{in } \Omega \\ \mathrm{tr}\ \tilde{\phi}^{(0)} = 0 & \text{on } \Gamma \\ \mathrm{tr}\ \tilde{\phi}^{(0)} = \tilde{\lambda}^{(0)} & \text{on } \Gamma' \end{cases}$$

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Hybrid weak formulation : *Given* $f^{(n)} \in L^2\Lambda^n(\Omega)$, *seek*

$$\left\{u^{(n-1)}, \tilde{\phi}^{(0)}, \tilde{\lambda}^{(0)}\right\} \in \left\{H\Lambda^{n-1}(\Omega), L^2\tilde{\Lambda}^0(\Omega), H_0^{1/2}\tilde{\Lambda}^0(\Gamma)\right\},$$

such that

$$\begin{cases} \left(v^{(n-1)}, u^{(n-1)}\right)_\Omega - \int_\Omega \mathrm{d}v^{(n-1)} \wedge \tilde{\phi}^{(0)} \pm \int_{\Gamma'} \tilde{\lambda}^{(0)} \wedge \mathrm{tr}\ v^{(n-1)} & = 0 \\ -\int_\Omega \mathrm{d}u^{(n-1)} \wedge \tilde{\psi}^{(0)} & = \int_\Omega f^{(n)} \wedge \tilde{\psi}^{(0)}\ , \\ \int_{\Gamma'} \tilde{\gamma}^{(0)} \wedge \left[\mathrm{tr}\ u^{(n-1)}\right] & = 0 \end{cases}$$

for all $\left\{v^{(n-1)}, \tilde{\psi}^{(0)}, \tilde{\gamma}^{(0)}\right\} \in \left\{H\Lambda^{(n-1)}(\Omega), L^2\tilde{\Lambda}^0(\Omega), H_0^{1/2}\tilde{\Lambda}(\Gamma)\right\}.$

Discretization

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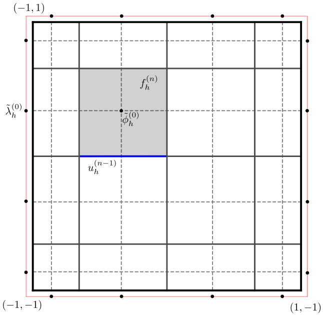

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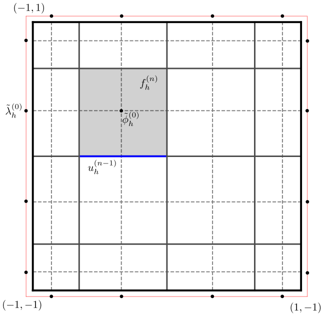

- $L_h^2\tilde{\Lambda}^0(\Omega_{\mathrm{ref}}) : P_{N,N}^{(\mathrm{G};0)}(\xi,\eta)$, the function space spanned by the polynomials based on Gauss nodes.
- $H_h\Lambda^{(n-1)}(\Omega_{\mathrm{ref}}) : L_{N,N}^{(\mathrm{GL};1)}(\xi,\eta)$, the function space spanned by the polynomials based on Gauss-Lobatto edges.
- $L_h^2\Lambda^{(n)}(\Omega_{\mathrm{ref}}) : S_{N,N}^{(\mathrm{GL};2)}(\xi,\eta)$, the function space spanned by the polynomials based on Gauss-Lobatto faces.
- $H_h^{1/2}\tilde{\Lambda}(\Gamma_{\mathrm{ref}}) : P_N^{(\mathrm{G};0)}(\xi)$, the function space spanned by the polynomials based on Gauss nodes in the 1-d reference space $\Gamma_{\mathrm{ref}}|\,\xi=[-1,1]$.

Discretization

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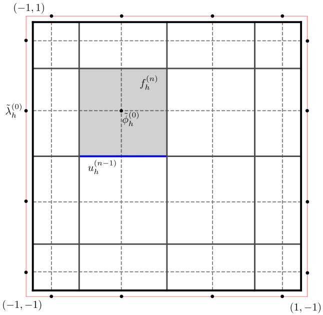

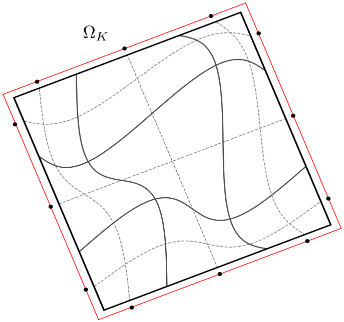

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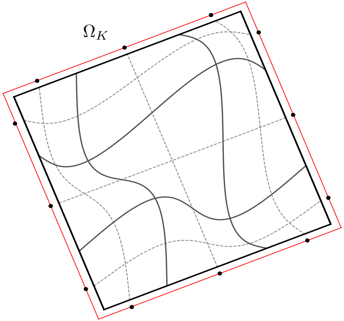
Mapping $\Phi_K:\ \Omega_{\mathrm{ref}}|\,(\xi,\eta)=[-1,1]^2\rightarrow\Omega_K|\,(x,y)$



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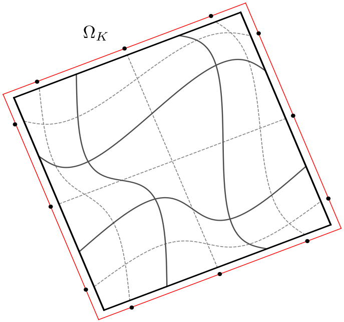
$$\begin{cases} \left(v^{(n-1)}, u^{(n-1)}\right)_\Omega - \int_\Omega \mathrm{d}v^{(n-1)} \wedge \tilde{\phi}^{(0)} \pm \int_{\Gamma'} \tilde{\lambda}^{(0)} \wedge \mathrm{tr}\, v^{(n-1)} & = 0 \\ -\int_\Omega \mathrm{d}u^{(n-1)} \wedge \tilde{\psi}^{(0)} & = \int_\Omega f^{(n)} \wedge \tilde{\psi}^{(0)} \\ \int_{\Gamma'} \tilde{\gamma}^{(0)} \wedge \left[\mathrm{tr}\, u^{(n-1)}\right] & = 0 \end{cases}$$

Mapping $\Phi_K : \Omega_{\mathrm{ref}} | (\xi, \eta) = [-1,1]^2 \to \Omega_K | (x,y)$

Pullback : $\Phi_K^* : \Lambda^k(\Phi(\Omega_K)) \to \Lambda^k(\Omega_{\mathrm{ref}})$

$$\int_{\Phi(\mathcal{M})} a^{(k)} = \int_{\mathcal{M}} \Phi^* a^{(k)}.$$



Discretization

$$\left\{u^{(n-1)}, \tilde{\phi}^{(0)}, \tilde{\lambda}^{(0)}\right\} \in \left\{H\Lambda^{n-1}(\Omega), L^2\tilde{\Lambda}^0(\Omega), H_0^{1/2}\tilde{\Lambda}^0(\Gamma)\right\},$$

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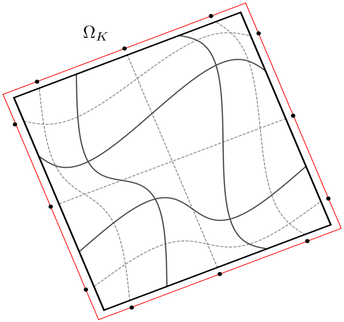
Pullback : $\Phi_K^* : \Lambda^k(\Phi(\Omega_K)) \rightarrow \Lambda^k(\Omega_{\mathrm{ref}})$

$$\int_{\Phi(\mathcal{M})} a^{(k)} = \int_{\mathcal{M}} \Phi^* a^{(k)}.$$

Pullback commutes with the metric-independent operators, i.e. exterior derivative d and the wedge product $\wedge$:

$$\Phi^* \mathrm{d}a^{(k)} = \mathrm{d}\Phi^* a^{(k)}$$

$$\Phi^* \left(a^{(k)} \wedge b^{(l)}\right) = \Phi^* a^{(k)} \wedge \Phi^* b^{(l)}$$



Discretization

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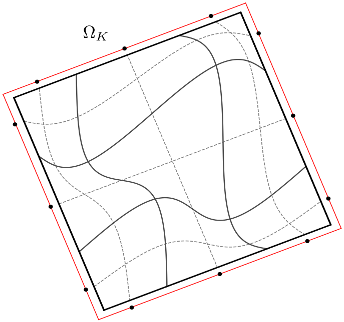
Basis functions in Ω_K are defined by the **inverse pullback** of the basis function in Ω_{ref} : $\Phi_K^{-*}(\cdot)$



Discretization

$$\left\{u^{(n-1)},\tilde{\phi}^{(0)},\tilde{\lambda}^{(0)}\right\}\in\left\{H\Lambda^{n-1}(\Omega),L^2\tilde{\Lambda}^0(\Omega),H_0^{1/2}\tilde{\Lambda}^0(\Gamma)\right\},$$

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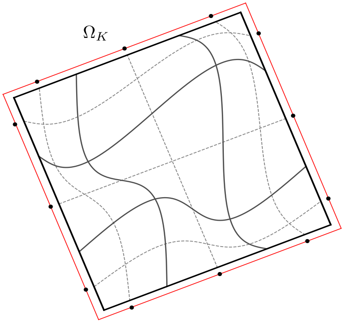
$$\int_{\Omega_K}\mathrm{d}u^{(n-1)}\wedge\tilde{\psi}^{(0)}=$$



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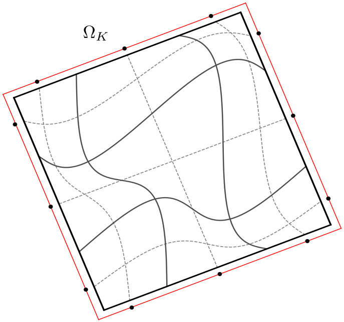
$$\int_{\Omega_K}\mathrm{d}u^{(n-1)}\wedge\tilde{\psi}^{(0)}=\int_{\Phi_K(\Omega_{\mathrm{ref}})}\mathrm{d}\Phi_K^{-*}u_{\mathrm{ref}}^{(n-1)}\wedge\Phi_K^{-*}\tilde{\psi}_{\mathrm{ref}}^{(0)}$$



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$$\left\{u^{(n-1)},\tilde{\phi}^{(0)},\tilde{\lambda}^{(0)}\right\}\in\left\{H\Lambda^{n-1}(\Omega),L^2\tilde{\Lambda}^0(\Omega),H_0^{1/2}\tilde{\Lambda}^0(\Gamma)\right\},$$

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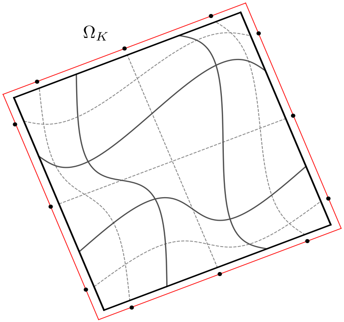
$$\int_{\Omega_K}\mathrm{d}u^{(n-1)}\wedge\tilde{\psi}^{(0)}=\int_{\Phi_K(\Omega_{\text{ref}})}\mathrm{d}\Phi_K^{-*}u_{\text{ref}}^{(n-1)}\wedge\Phi_K^{-*}\tilde{\psi}_{\text{ref}}^{(0)}$$

$$=\int_{\Omega_{\text{ref}}}\Phi_K^{*}\left(\mathrm{d}\Phi_K^{-*}u_{\text{ref}}^{(n-1)}\wedge\Phi_K^{-*}\tilde{\psi}_{\text{ref}}^{(0)}\right)$$



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$$\left\{u^{(n-1)}, \tilde{\phi}^{(0)}, \tilde{\lambda}^{(0)}\right\} \in \left\{H\Lambda^{n-1}(\Omega), L^2\tilde{\Lambda}^0(\Omega), H_0^{1/2}\tilde{\Lambda}^0(\Gamma)\right\},$$

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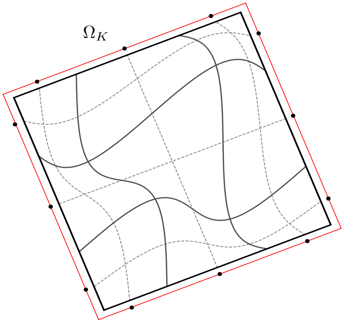

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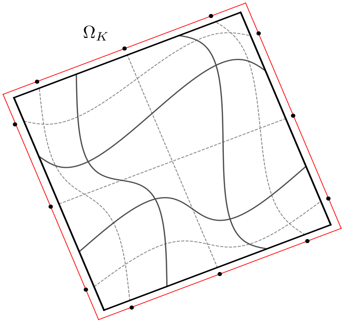
In discrete form, it reads $\mathbb{W}\mathbb{E}^{n,n-1}\boldsymbol{u}_K$



Discretization

$$\begin{bmatrix} A_K & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} u_K \\ \phi_K \\ \lambda_K \end{bmatrix} = \begin{bmatrix} 0 \\ Wf_K \\ 0 \end{bmatrix}$$

Basis functions in Ω_K are defined by the **inverse pullback** of the basis function in Ω_{ref} : $\Phi_K^{-*}(\cdot)$

$$\begin{aligned} \int_{\Omega_K} \mathrm{d}u^{(n-1)} \wedge \tilde{\psi}^{(0)} &= \int_{\Phi_K(\Omega_{\mathrm{ref}})} \mathrm{d}\Phi_K^{-*} u_{\mathrm{ref}}^{(n-1)} \wedge \Phi_K^{-*} \tilde{\psi}_{\mathrm{ref}}^{(0)} \\ &= \int_{\Omega_{\mathrm{ref}}} \Phi_K^{*} \left(\mathrm{d}\Phi_K^{-*} u_{\mathrm{ref}}^{(n-1)} \wedge \Phi_K^{-*} \tilde{\psi}_{\mathrm{ref}}^{(0)} \right) \\ &= \int_{\Omega_{\mathrm{ref}}} \mathrm{d}u_{\mathrm{ref}}^{(n-1)} \wedge \tilde{\psi}_{\mathrm{ref}}^{(0)} \end{aligned}$$

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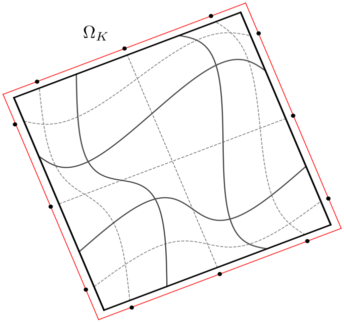

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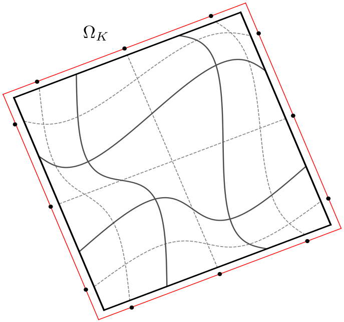
$$\begin{bmatrix} A_K & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} u_K \\ \phi_K \\ \lambda_K \end{bmatrix} = \begin{bmatrix} 0 \\ Wf_K \\ 0 \end{bmatrix}$$

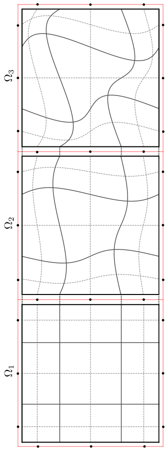
- $A_K = M_K^{(n-1)}$: the symmetric mass matrix, changes from element to element.
- $B = W\mathbb{E}^{n,n-1}$: same in every element.
- C : wedge matrix related to the weak continuity term

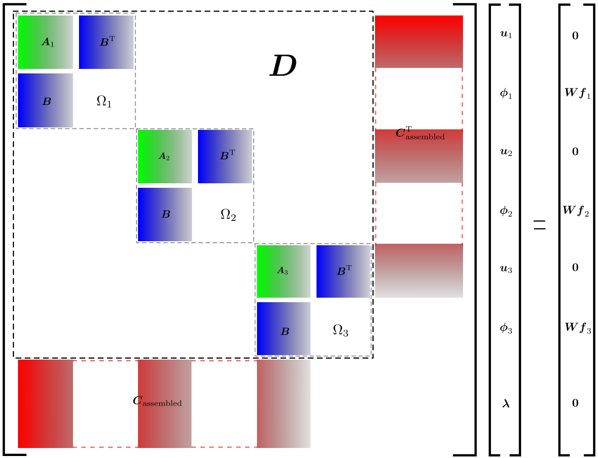
$$\int_{\Gamma'} \tilde{\lambda}_h^{(0)} \wedge \left[\text{tr}\, u_h^{(n-1)} \right]$$

same in every element.



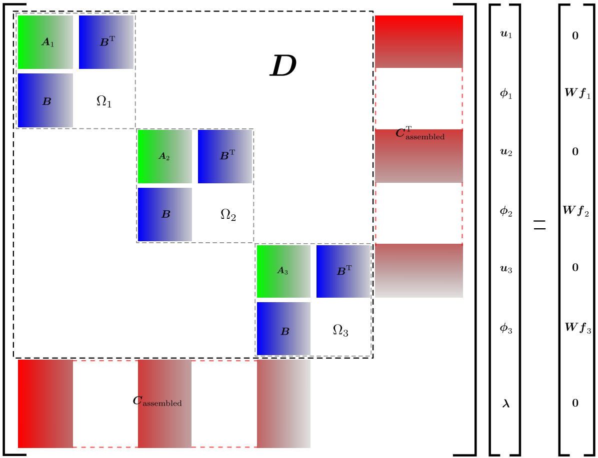
Global system





Global system

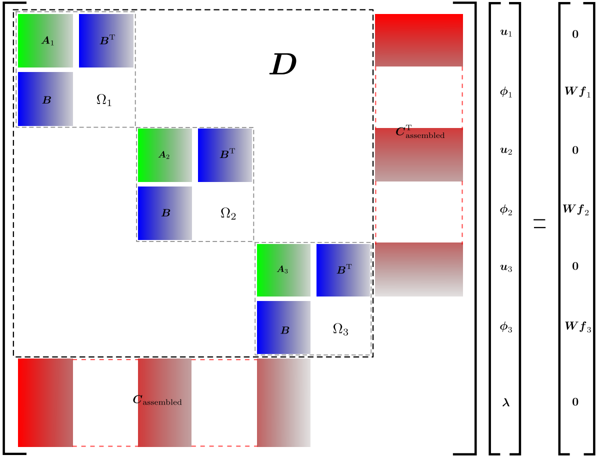
$$
\begin{bmatrix} D_K & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix}
$$



Global system

$$\begin{bmatrix} D_K & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix}$$

$$CD^{-1}C^{\mathrm{T}}\lambda = CD^{-1}h$$

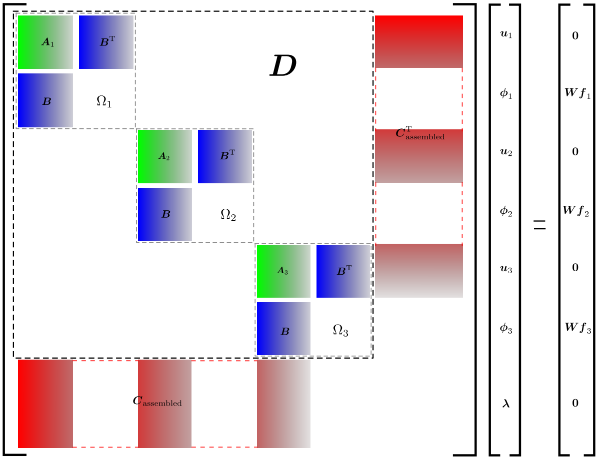


Global system

$$\begin{bmatrix} D_K & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix}$$

$$CD^{-1}C^{\mathrm{T}}\lambda = CD^{-1}h$$

After solving for λ, the remaining three local problems are trivial since D^{-1} is already there.



Analytical solution

$$\tilde{\phi}^{(0)}_{\text{exact}} = \sin(2\pi x)\sin(2\pi y)$$

$$\Omega |\, (x, y) = [0, 1]^2$$

$$\begin{cases} x = \frac{1}{2} + \frac{1}{2}\left(\xi + c\sin(\pi\xi)\sin(\pi\eta)\right) \\ y = \frac{1}{2} + \frac{1}{2}\left(\eta + c\sin(\pi\xi)\sin(\pi\eta)\right) \end{cases}$$

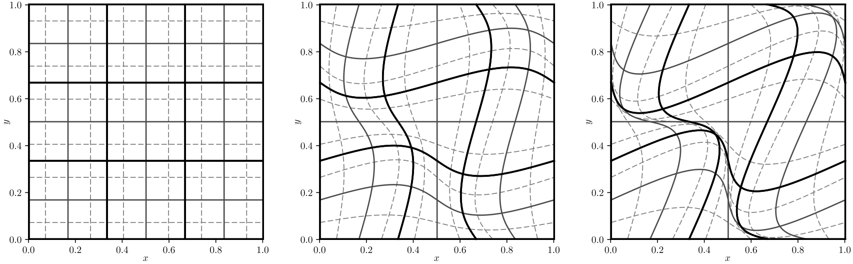
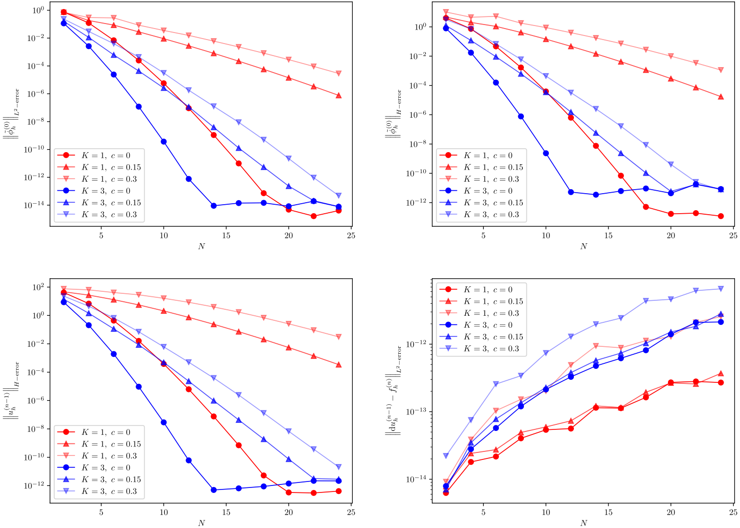
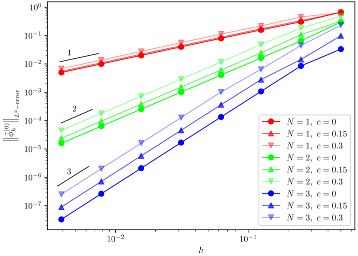


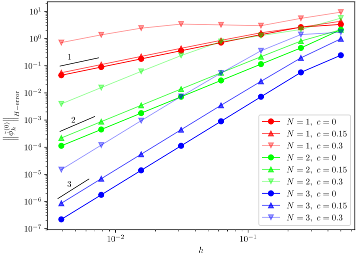
FIGURE – 3×3 elements, polynomial order $N = 2$. Left : $c = 0$. Middle : $c = 0.15$. Right : $c = 0.3$

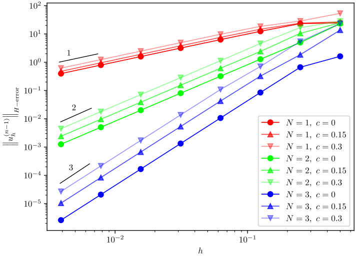
Analytical solution, *ph*-convergence

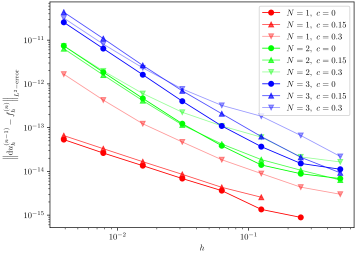


Analytical solution, *ph*-convergence

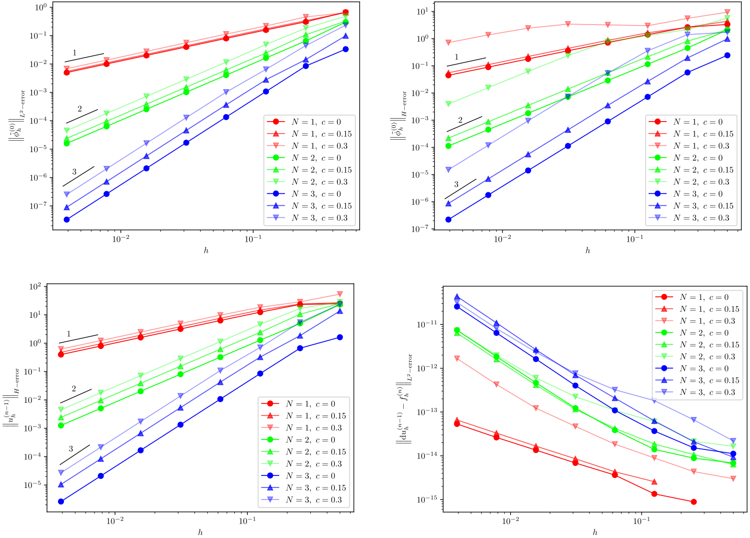






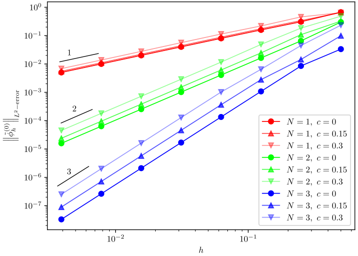


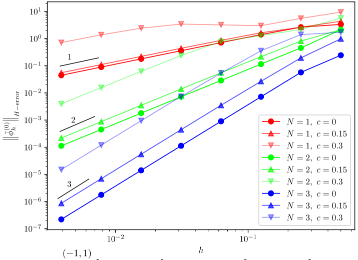
Analytical solution, *ph*-convergence



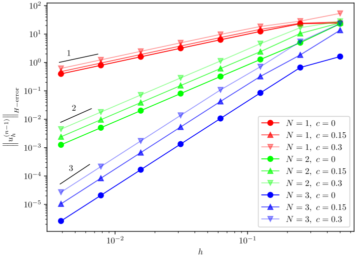
super-convergence ?

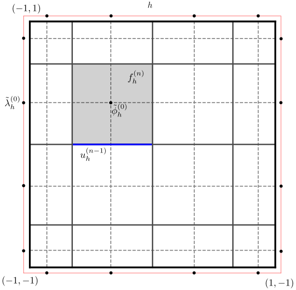
Analytical solution, *ph*-convergence



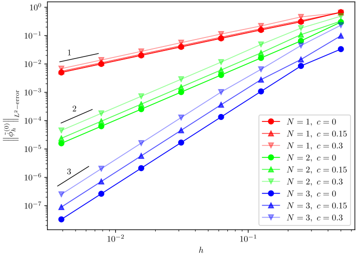


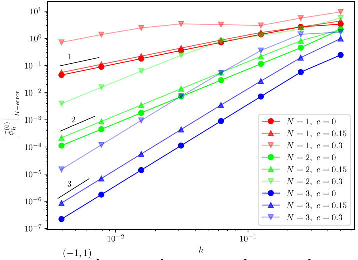
super-convergence ?





Analytical solution, *ph*-convergence





super-convergence ?

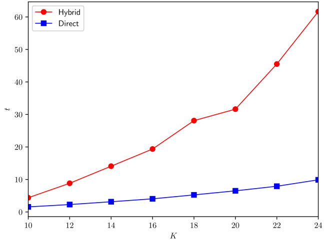
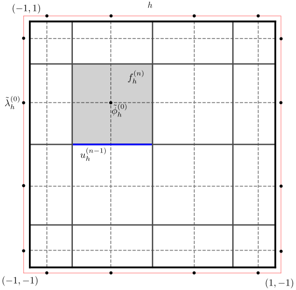
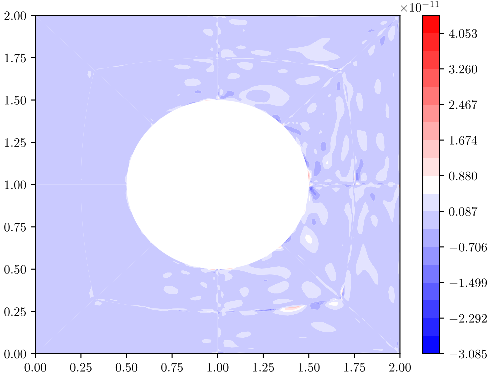


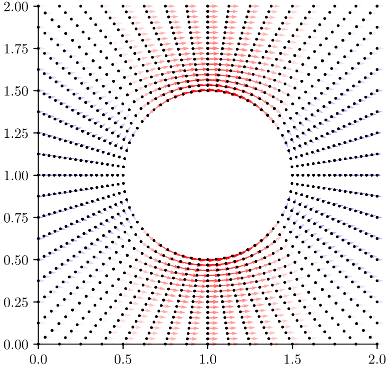
FIGURE – Time consumption at $N = 10$, $K \times K$ elements



2-d potential flow around cylinder



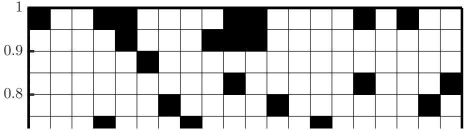
(a) $\mathrm{d}u_h^{(n-1)} - f_h^{(n)}$

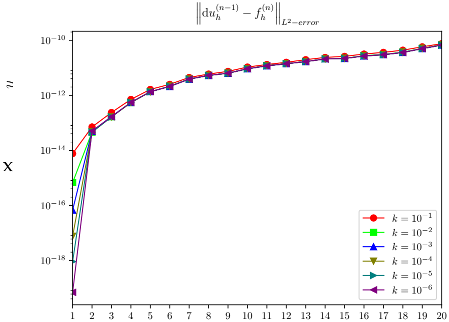


(b) $u_h^{(n-1)}$

FIGURE – 2-d potential flow around cylinder.

The Sand-Shale problem





(a) Sand Shale domain

TABLE – Net flux through the left or right boundary at polynomial order $N = 20$

$k = 10^{-1}$	$k = 10^{-2}$	$k = 10^{-3}$	$k = 10^{-4}$	$k = 10^{-5}$	$k = 10^{-6}$
0.676763	0.562508	0.526257	0.520749	0.520166	0.520107

Thanks

Thanks, questions ?