Hybrid mimetic spectral element methods with dual polynomials

With applications in Poisson and linear elasticity

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Summary

Mimetic

Mimetic methods aim to preserve the structure of partial differential equations at the discrete level.

A key feature of mimetic mixed finite element methods is that their finite dimensional function spaces preserve the so-called De Rham complex :

$$
\begin{array}{ccccccccc}
\mathbb{R} \rightarrow H^1 & \overset{\text{grad}}{\rightarrow} & H(\text{curl}) & \overset{\text{curl}}{\rightarrow} & H(\text{div}) & \overset{\text{div}}{\rightarrow} & L^2 \rightarrow 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R} \rightarrow H_h^1 & \overset{\text{grad}}{\rightarrow} & H_h(\text{curl}) & \overset{\text{curl}}{\rightarrow} & H_h(\text{div}) & \overset{\text{div}}{\rightarrow} & L_h^2 \rightarrow 0.
\end{array}
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Therefore, mimetic methods are also called structure-preserving methods.

Mimetic Spectral Element Method [1, 2, 3] is a high order mimetic mixed finite element method using the mathematical language of differential geometry and algebraic topology.

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Dual

Duality structure is very common in physics.

- Fluid : pressure p and source term f, div $\underline{u}$.
- Elasticity : displacement $\underline{u}$ and body force $\underline{f}$, div $\underline{\underline{\sigma}}$.

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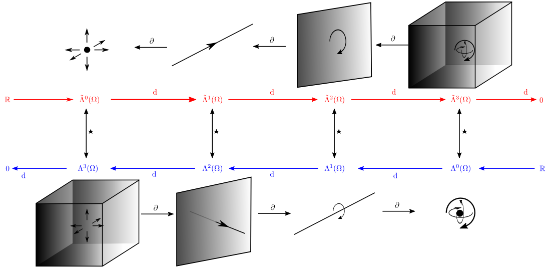

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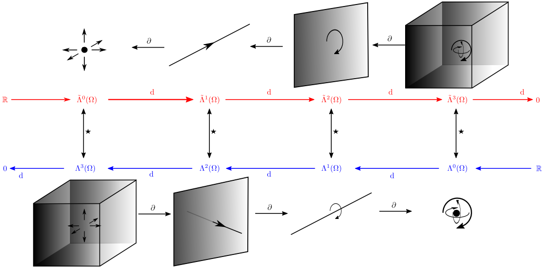
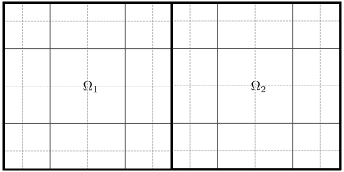
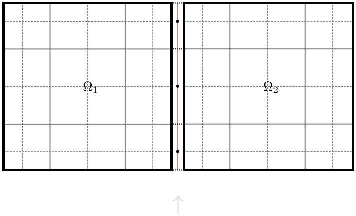


The operator duality pairing between vectors from two dual spaces is well-defined and independent of metric. We would like to preserve this duality structure.

Hybrid

Hybrid (finite element) methods are those methods that relax the continuity across the inter-element interface by introducing a Lagrange multiplier between elements.





For more information about hybrid methods, we refer to Pian [4], Raviart and Thomas [5], Brezzi and Fortin [6].

4. Pian, T.H. Derivation of element stiffness matrices by assumed stress distributions. *AIAA journal*, (1964) 2(7), 1333-1336.
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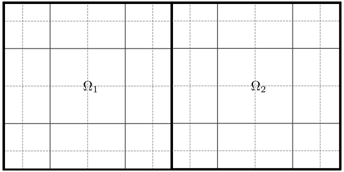

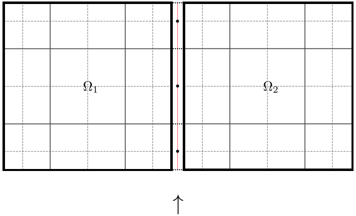

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Mimetic basis functions

Let $-1 = \xi_0 < \xi_1 < \cdots < \xi_N = 1$ be a partitioning of the interval $[-1, 1]$. The associated Lagrange polynomials :

$$h_i(\xi),\ \xi \in [-1, 1],\ i = 0, 1, \cdots, N,\ \text{satisfying}\ h_i(\xi_j) = \delta_{i,j}\ \text{(Kronecker delta)}.$$

The corresponding edge polynomials[7] are

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{\mathrm{d}h_k(\xi)}{\mathrm{d}\xi} = \sum_{k=i}^{N} \frac{\mathrm{d}h_k(\xi)}{\mathrm{d}\xi},\ i = 1, 2, \cdots, N,\ \text{satisfying}\ \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j}.$$

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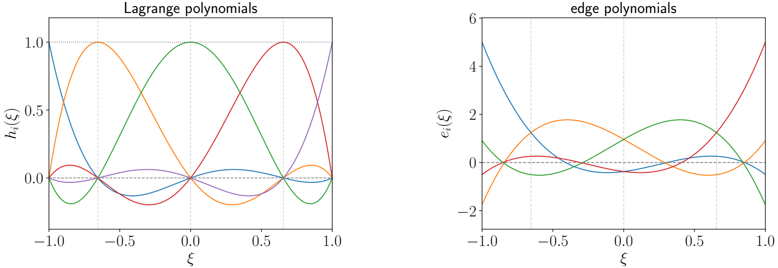

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Finite dimensional spaces spanned by $\left\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\right\}$ and $\left\{e_i(\xi)e_j(\eta)\right\}$ satisfy the De Rham complex. Let $\boldsymbol{u}, f$ be expanded as

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta)\right) \quad \text{and} \quad f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \mathrm{div}\ \boldsymbol{u}$, then $f_h = \mathrm{div}\ \boldsymbol{u}_h$ and

$$f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}\right) e_i(\xi) e_j(\eta) = \mathrm{div}\ \boldsymbol{u}_h.$$

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Discrete divergence operator

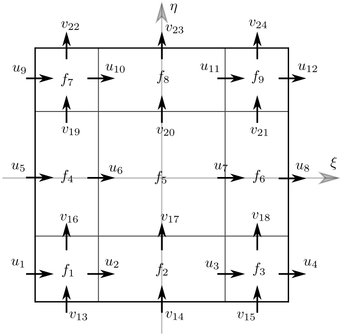

FIGURE – Reference domain.

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right),$$

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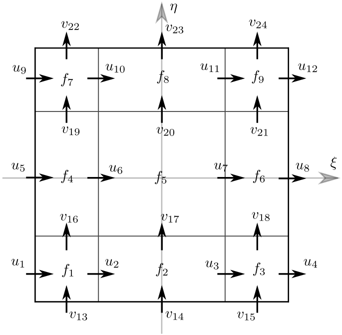

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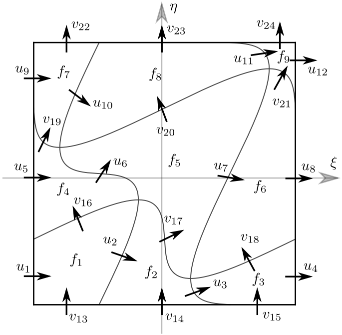

FIGURE – Curvilinear domain.

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N} \sum_{j=1}^{N} u'_{i,j}\ a_{i,j}(\xi,\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v'_{i,j}\ b_{i,j}(\xi,\eta) \right),$$

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Discrete trace operator

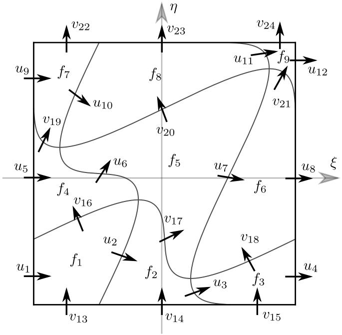

FIGURE – Curvilinear domain.

The trace variable $\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}$ can be discretized as

$$\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}_h = \left\{ \sum_{i=1}^{N} v_i^{\mathsf{s}} e_i'(\xi),\ \sum_{i=1}^{N} v_i^{\mathsf{n}} e_i'(\xi),\ \sum_{i=1}^{N} u_i^{\mathsf{w}} e_i'(\eta),\ \sum_{i=1}^{N} u_i^{\mathsf{e}} e_i'(\eta) \right\}.$$

There is a linear operator, $\mathbb{N}$, such that

$$\boldsymbol{u}_{\mathrm{tr}}' = \mathbb{N}\boldsymbol{u}',$$

where $\boldsymbol{u}_{\mathrm{tr}}' = \left(-v_i^{\mathsf{s}}, v_i^{\mathsf{n}}, -u_i^{\mathsf{w}}, u_i^{\mathsf{e}}\right)^T$ and

$$\mathbb{N} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
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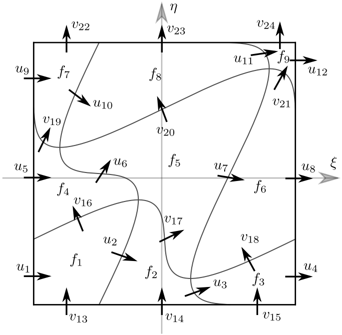

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There is a linear operator, $\mathbb{N}$, such that

$$\boldsymbol{u}'_{\mathrm{tr}} = \mathbb{N}\boldsymbol{u}',$$

where $\boldsymbol{u}'_{\mathrm{tr}} = \left(-\boldsymbol{v}_i^{\mathsf{s}}, \boldsymbol{v}_i^{\mathsf{n}}, -\boldsymbol{u}_i^{\mathsf{w}}, \boldsymbol{u}_i^{\mathsf{e}}\right)^T$ and

$$\mathbb{N} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

$\mathbb{N}$ is the discrete trace operator.

Dual representations

Let scalar functions p_h and q_h both be expanded in terms of basis functions $\{e_i(\xi)e_j(\eta)\}$,

$$(p_h, q_h)_{L^2(\Omega)} = \boldsymbol{p}^T \mathbb{M}^{(2)} \boldsymbol{q},$$

where $\mathbb{M}^{(2)}$ is the mass matrix. We can further define the dual basis functions[8] as

$$\left[\widetilde{e_1(\xi)e_1(\eta)}, \cdots, \widetilde{e_N(\xi)e_N(\eta)}\right] := \left[e_1(\xi)e_1(\eta), \cdots, e_N(\xi)e_N(\eta)\right] {\mathbb{M}^{(2)}}^{-1}.$$

8. Jain, V., Zhang, Y., Palha, A. and Gerritsma, M. Construction and application of algebraic dual polynomial representations for finite element methods. (2017) arXiv :1712.09472.

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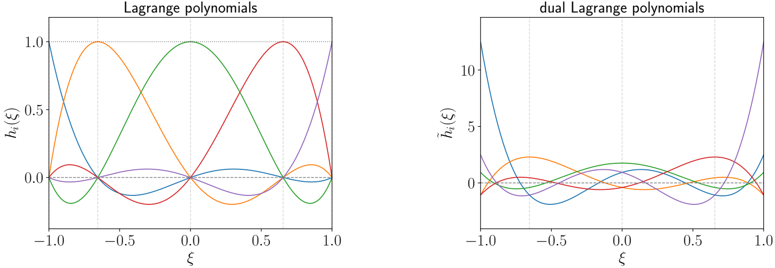

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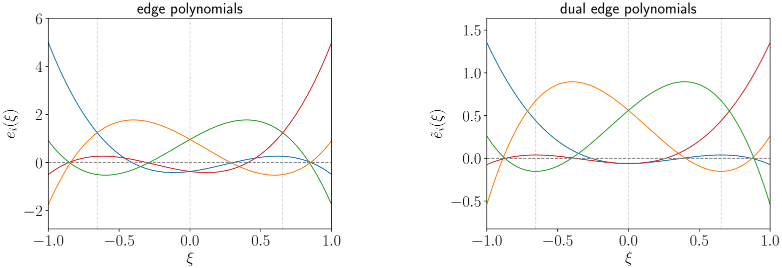

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If we expand p in terms of the dual basis functions $\left\{e_i\widetilde{(\xi)e_j}(\eta)\right\}$, we can obtain

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \tilde{\boldsymbol{p}}^T \boldsymbol{q}, \text{ where } \tilde{\boldsymbol{p}} = \mathbb{M}^{(2)} \boldsymbol{p},$$

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \langle \mathcal{R} p_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = (p_h, q_h)_{L^2(\Omega)}.$$

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$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \langle \mathcal{R} p_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = (p_h, q_h)_{L^2(\Omega)} .$$

Riesz Representation Theorem : *For every $\tilde{\boldsymbol{u}} \in \tilde{V}$, there exists a unique $\boldsymbol{u} \in V$, such that*

$$\langle \tilde{\boldsymbol{u}}, \boldsymbol{v} \rangle_{\tilde{V}\times V} = \langle \mathcal{R}\boldsymbol{u}, \boldsymbol{v} \rangle_{\tilde{V}\times V} = (\boldsymbol{u}, \boldsymbol{v})_V, \forall \boldsymbol{v} \in V,$$

$\mathcal{R} : \boldsymbol{u} \in V \rightarrow \tilde{\boldsymbol{u}} \in \tilde{V}$ is called Riesz mapping.

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If we expand p in terms of the dual basis functions $\left\{e_i\widetilde{(\xi)e_j}(\eta)\right\}$, we can obtain

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \tilde{\boldsymbol{p}}^T \boldsymbol{q}, \text{ where } \tilde{\boldsymbol{p}} = \mathbb{M}^{(2)} \boldsymbol{p},$$

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \langle \mathcal{R} p_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = (p_h, q_h)_{L^2(\Omega)}.$$

Furthermore, if $q_h = \text{div } \boldsymbol{v}_h$, and $\boldsymbol{v}_h$ is expanded by basis functions $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\}$, we have

$$\langle \tilde{p}_h, \text{div } \boldsymbol{v}_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \tilde{\boldsymbol{p}}^T \mathbb{E}^{2,1} \boldsymbol{v}.$$

The same idea can be applied to the trace basis functions.

8. Jain, V., Zhang, Y., Palha, A. and Gerritsma, M. Construction and application of algebraic dual polynomial representations for finite element methods. (2017) arXiv :1712.09472.

Sobolev spaces

Given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, let $L^2(\Omega)$ be the space of square integrable scalar-valued functions in Ω,

$$L^2(\Omega) := \left\{ \varphi \,\middle|\, (\varphi, \varphi)_{L^2(\Omega)} = \int_\Omega |\varphi|^2 \,\mathrm{d}\Omega < +\infty \right\},$$

then,

$$H^1(\Omega) := \left\{ \varphi \in L^2(\Omega) \,\middle|\, \mathrm{grad}\ \varphi \in \left[L^2(\Omega)\right]^d \right\},$$

$$H(\mathrm{div}, \Omega) := \left\{ \underline{u} \in \left[L^2(\Omega)\right]^d \,\middle|\, \mathrm{div}\ \underline{u} \in L^2(\Omega) \right\}.$$

And the trace spaces are defined as

$$H^{1/2}(\partial\Omega) := \mathrm{tr}_{\mathrm{grad}} H^1(\Omega), \quad H^{-1/2}(\partial\Omega) := \mathrm{tr}_{\mathrm{div}} H(\mathrm{div}, \Omega),$$

which form a pair of dual spaces.

Broken Sobolev spaces

Given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. A mesh, denoted by Ω^h, partitions Ω into K disjoint open elements Ω_k with Lipschitz boundary $\partial\Omega_k$,

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k, \ \Omega_i \cap \Omega_j = \emptyset, \ 1 \leq i \neq j \leq K.$$

We can break $L^2(\Omega)$, $H^1(\Omega)$, $H(\mathrm{div}, \Omega)$ and obtain the so-called broken Sobolev spaces [9] :

$$L^2(\Omega^h) = \left\{ \varphi \in L^2(\Omega) \middle| \ \varphi|_{\Omega_k} \in L^2(\Omega_k) \right\} = \prod_{k=1}^{K} L^2(\Omega_k),$$

$$H^1(\Omega^h) = \left\{ \varphi \in L^2(\Omega) \middle| \ \varphi|_{\Omega_k} \in H^1(\Omega_k) \right\} = \prod_{k=1}^{K} H^1(\Omega_k),$$

$$H(\mathrm{div}, \Omega^h) = \left\{ \boldsymbol{u} \in \left[L^2(\Omega)\right]^d \middle| \ \boldsymbol{u}|_{\Omega_k} \in H(\mathrm{div}, \Omega_k) \right\} = \prod_{k=1}^{K} H(\mathrm{div}, \Omega_k).$$

Spaces for interface functions are then defined as

$$H^{1/2}(\partial\Omega^h) := \mathrm{tr}^h_{\mathrm{grad}} H^1(\Omega), \qquad H^{-1/2}(\partial\Omega^h) := \mathrm{tr}^h_{\mathrm{div}} H(\mathrm{div}, \Omega),$$

which are a pair of dual spaces as well. $\mathrm{tr}^h_{\mathrm{grad}}$, $\mathrm{tr}^h_{\mathrm{div}}$ restrict $\varphi \in H^1(\Omega)$, $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ onto $\partial\Omega_h = \bigcup_{k=1}^{K} \partial\Omega_k$.

9. Carstensen, C., Demkowicz, L. and Gopalakrishnan, J. Breaking spaces and forms for the DPG method and applications including Maxwell equations. *Computers and Mathematics with Applications*, (2016) 72(3) : 494-522.

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Poisson problem

Mixed formulation

We consider the constrained minimization problem,

$$\underset{\boldsymbol{u}\in L^2(\Omega)}{\arg} \min_{\mathrm{div}\boldsymbol{u}=-f} \frac{1}{2}\left(\boldsymbol{u},\boldsymbol{u}\right)_{L^2(\Omega)},$$

where f is given. By introducing a Lagrange multiplier φ, we can rewrite this constrained minimization problem into a saddle-point problem for $(\boldsymbol{u},\varphi)\in H(\mathrm{div},\Omega)\times\tilde{L}^2(\Omega)$:

$$\mathcal{L}(\boldsymbol{u},\varphi;f,\hat{\varphi})=\frac{1}{2}\left(\boldsymbol{u},\boldsymbol{u}\right)_{L^2(\Omega)}+\left\langle\varphi,\mathrm{div}\ \boldsymbol{u}+f\right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)}-\left\langle\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\right\rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)},$$

where $\hat{\varphi}=\mathrm{tr}_{\mathrm{grad}}\ \varphi\in H^{1/2}(\partial\Omega)$ and $f\in L^2(\Omega)$ is given.

Variational analysis on this functional gives rise to the mixed formulation : *Find* $(\boldsymbol{u},\varphi)\in H(\mathrm{div},\Omega)\times\tilde{L}^2(\Omega)$ *such that*

$$\begin{cases}\left(\boldsymbol{u},\boldsymbol{v}\right)_{L^2(\Omega)}+\left\langle\varphi,\mathrm{div}\ \boldsymbol{v}\right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} & =\left\langle\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\right\rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \\ \left\langle\psi,\mathrm{div}\ \boldsymbol{u}\right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} & =-\left\langle\psi,f\right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)}\end{cases},$$

for all $(\boldsymbol{v},\psi)\in H(\mathrm{div},\Omega)\times\tilde{L}^2(\Omega)$.

This is a weak mixed formulation of the Poisson equation.

Mixed formulation

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Variational analysis on this functional gives rise to the mixed formulation : *Find* $(\boldsymbol{u},\varphi)\in H(\mathrm{div},\Omega)\times\tilde{L}^2(\Omega)$ *such that*

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for all $(\boldsymbol{v},\psi)\in H(\mathrm{div},\Omega)\times\tilde{L}^2(\Omega)$.

This is a weak mixed formulation of the Poisson equation.

Hybrid mixed formulation

If we set up a mesh Ω^h in Ω, using broken spaces, $L^2(\Omega^h)$, $H(\mathrm{div},\Omega^h)$ and $H^1(\Omega^h)$, and introducing a new Lagrange multiplier $\check{\varphi}$ in the interface space $H^{1/2}(\partial\Omega^h\backslash\partial\Omega)$, we can rewrite the functional as

$$\mathcal{L}(\boldsymbol{u},\varphi,\check{\varphi};f,\hat{\varphi}) = \frac{1}{2}\left(\boldsymbol{u},\boldsymbol{u}\right)_{L^2(\Omega^h)} + \langle\varphi,\mathrm{div}\ \boldsymbol{u}+f\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \langle\check{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} - \left(\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\right)_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}.$$

The interface variable $\check{\varphi}$ serves as the Lagrange multiplier which enforces the continuity at the internal interface $\partial\Omega_h\backslash\partial\Omega$.

From this new functional, we can obtain the hybrid mixed formulation for the Poisson problem written as : *Given* $f\in L^2(\Omega^h)$ *and* $\hat{\varphi}=\mathrm{tr}_{\mathrm{grad}}\ \varphi\in H^{1/2}(\partial\Omega)$*, find* $(\boldsymbol{u},\varphi,\check{\varphi})\in H(\mathrm{div},\Omega^h)\times\tilde{L}^2(\Omega^h)\times H^{1/2}(\partial\Omega^h\backslash\partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega^h)} + \langle\varphi,\mathrm{div}\ \boldsymbol{v}\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \langle\check{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \langle\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \\ \langle\psi,\mathrm{div}\ \boldsymbol{u}\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} & = -\langle\psi,f\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\ -\langle\check{\psi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v},\psi,\check{\psi})\in H(\mathrm{div},\Omega^h)\times\tilde{L}^2(\Omega^h)\times H^{1/2}(\partial\Omega^h\backslash\partial\Omega)$. It is easy to prove that the interface variable $\check{\varphi}$ represents the restriction of φ onto $\partial\Omega^h\backslash\partial\Omega$.

Hybrid mixed formulation

If we set up a mesh Ω^h in Ω, using broken spaces, $L^2(\Omega^h)$, $H(\mathrm{div},\Omega^h)$ and $H^1(\Omega^h)$, and introducing a new Lagrange multiplier $\check{\varphi}$ in the interface space $H^{1/2}(\partial\Omega^h\backslash\partial\Omega)$, we can rewrite the functional as

$$\mathcal{L}(\boldsymbol{u},\varphi,\check{\varphi};f,\hat{\varphi}) = \frac{1}{2}\left(\boldsymbol{u},\boldsymbol{u}\right)_{L^2(\Omega^h)} + \left\langle \varphi, \mathrm{div}\ \boldsymbol{u} + f\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \left\langle \check{\varphi}, \mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\right\rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} - \left(\hat{\varphi}, \mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\right)_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}.$$

The interface variable $\check{\varphi}$ serves as the Lagrange multiplier which enforces the continuity at the internal interface $\partial\Omega_h\backslash\partial\Omega$.

From this new functional, we can obtain the hybrid mixed formulation for the Poisson problem written as : *Given $f \in L^2(\Omega^h)$ and $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\ \varphi \in H^{1/2}(\partial\Omega)$, find $(\boldsymbol{u},\varphi,\check{\varphi}) \in H(\mathrm{div},\Omega^h)\times\tilde{L}^2(\Omega^h)\times H^{1/2}(\partial\Omega^h\backslash\partial\Omega)$ such that*

$$\begin{cases} \left(\boldsymbol{u},\boldsymbol{v}\right)_{L^2(\Omega^h)} + \left\langle\varphi,\mathrm{div}\ \boldsymbol{v}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \left\langle\check{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\right\rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} &= \left\langle\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\right\rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \\ \left\langle\psi,\mathrm{div}\ \boldsymbol{u}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} &= -\left\langle\psi,f\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\ -\left\langle\check{\psi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\right\rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} &= 0 \end{cases},$$

for all $(\boldsymbol{v},\psi,\check{\psi}) \in H(\mathrm{div},\Omega^h)\times\tilde{L}^2(\Omega^h)\times H^{1/2}(\partial\Omega^h\backslash\partial\Omega)$. It is easy to prove that the interface variable $\check{\varphi}$ represents the restriction of φ onto $\partial\Omega^h\backslash\partial\Omega$.

Discretization

Hybrid mixed formulation

Given $f \in L^2(\Omega^h)$ *and* $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\, \varphi \in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega^h)} + \langle \varphi, \mathrm{div}\, \boldsymbol{v} \rangle_{\tilde{L}^2(\Omega^h) \times L^2(\Omega^h)} - \langle \check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega^h \backslash \partial\Omega) \times H^{-1/2}(\partial\Omega^h \backslash \partial\Omega)} & = \langle \hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)} \\ \langle \psi, \mathrm{div}\, \boldsymbol{u} \rangle_{\tilde{L}^2(\Omega^h) \times L^2(\Omega^h)} & = - \langle \psi, f \rangle_{\tilde{L}^2(\Omega^h) \times L^2(\Omega^h)} \\ - \langle \check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u} \rangle_{H^{1/2}(\partial\Omega^h \backslash \partial\Omega) \times H^{-1/2}(\partial\Omega^h \backslash \partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$.

We choose finite dimensional spaces spanned by following basis functions for the discretization :

- $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\} \to H(\mathrm{div}, \Omega_k)$.
- $\{e_i(\xi)e_j(\eta)\} \to L^2(\Omega_k)$, $\left\{\widetilde{e_i(\xi)e_j(\eta)}\right\} \to \tilde{L}^2(\Omega_k)$.
- $\{e_i(s)\} \to H^{-1/2}(\partial\Omega_k)$, $\left\{\widetilde{e_i(s)}\right\} \to H^{1/2}(\partial\Omega_k)$.

Discretization

Hybrid mixed formulation

Given $f \in L^2(\Omega^h)$ *and* $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\, \varphi \in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega^h)} + \langle \varphi, \mathrm{div}\ \boldsymbol{v} \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \langle \check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \langle \hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \\ \langle \psi, \mathrm{div}\ \boldsymbol{u} \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} & = -\langle \psi, f \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\ -\langle \check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u} \rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$.

We choose finite dimensional spaces spanned by following basis functions for the discretization :

- $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\} \rightarrow H(\mathrm{div}, \Omega_k)$.
- $\{e_i(\xi)e_j(\eta)\} \rightarrow L^2(\Omega_k)$, $\left\{e_i\widetilde{(\xi)e_j}(\eta)\right\} \rightarrow \tilde{L}^2(\Omega_k)$.
- $\{e_i(s)\} \rightarrow H^{-1/2}(\partial\Omega_k)$, $\left\{\widetilde{e_i(s)}\right\} \rightarrow H^{1/2}(\partial\Omega_k)$.

Discretization

Hybrid mixed formulation

Given $f \in L^2(\Omega^h)$ *and* $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\, \varphi \in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega^h)} + \langle \varphi, \mathrm{div}\ \boldsymbol{v} \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \langle \check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \langle \hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \\ \langle \psi, \mathrm{div}\ \boldsymbol{u} \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} & = -\langle \psi, f \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\ -\langle \check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u} \rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$.

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

- $\mathbb{M}^{(1)}$: metric-dependent ; element-wise-block-diagonal ;
- $\mathbb{E}^{2,1}$: metric-independent ; element-wise-block-diagonal ; super sparse ; ± 1 non-zero entries ;
- $\mathbb{N}$: metric-independent ; even more sparse ; ± 1 non-zero entries ;

Discretization

Hybrid mixed formulation

Given $f \in L^2(\Omega^h)$ *and* $\hat{\varphi} = \mathrm{tr}_{\mathrm{grad}}\, \varphi \in H^{1/2}(\partial\Omega)$*, find* $(\boldsymbol{u}, \varphi, \check{\varphi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega^h)} + \langle \varphi, \mathrm{div}\ \boldsymbol{v} \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \langle \check{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \langle \hat{\varphi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{v} \rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \\ \langle \psi, \mathrm{div}\ \boldsymbol{u} \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} & = -\langle \psi, f \rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\ -\langle \check{\psi}, \mathrm{tr}_{\mathrm{div}} \boldsymbol{u} \rangle_{H^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times H^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

for all $(\boldsymbol{v}, \psi, \check{\psi}) \in H(\mathrm{div}, \Omega^h) \times \tilde{L}^2(\Omega^h) \times H^{1/2}(\partial\Omega^h \backslash \partial\Omega)$.

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

- $\mathbb{M}^{(1)}$: metric-dependent ; element-wise-block-diagonal ;
- $\mathbb{E}^{2,1}$: metric-independent ; element-wise-block-diagonal ; super sparse ; ± 1 non-zero entries ;
- $\mathbb{N}$: metric-independent ; even more sparse ; ± 1 non-zero entries ;

Discretization

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

- $\mathbb{M}^{(1)}$: metric-dependent ; element-wise-block-diagonal ;
- $\mathbb{E}^{2,1}$: metric-independent ; element-wise-block-diagonal ; super sparse ; ± 1 non-zero entries ;
- $\mathbb{N}$: metric-independent ; even more sparse ; ± 1 non-zero entries ;

We can easily eliminate $\underline{u}$ and $\underline{\varphi}$ and obtain a system for the discrete interface variable $\underline{\check{\varphi}}$,

$$\mathbb{H}\underline{\check{\varphi}} = \mathbb{F},$$

where

$$\mathbb{H} = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - {\mathbb{E}^{2,1}}^T \left(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \right)^{-1} \mathbb{E}^{2,1} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_I^T,$$

$$\mathbb{F} = \mathbb{F}_{\hat{\varphi}} + \mathbb{F}_f,$$

$$\mathbb{F}_{\hat{\varphi}} = \mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - {\mathbb{E}^{2,1}}^T \left(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \right)^{-1} \mathbb{E}^{2,1} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_B^T \underline{\hat{\varphi}},$$

$$\mathbb{F}_f = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \left(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \right)^{-1} \underline{f}.$$

- Inverting $\mathbb{M}^{(1)}$ and $\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T$ is easy (in parallel) because they are element-wise-block-diagonal.
- Solving for $\underline{\check{\varphi}}$ is cheap (smaller system size and condition number).
- Remaining local problems for $\underline{u}$ and $\underline{\varphi}$ are trivial because $(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T)^{-1}$ is already computed.

Discretization

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}^{(1)} & \mathbb{E}^{2,1^T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

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We can easily eliminate $\underline{\boldsymbol{u}}$ and $\underline{\varphi}$ and obtain a system for the discrete interface variable $\underline{\check{\varphi}}$,

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where

$$\mathbb{H} = -\mathbb{N}_I \mathbb{M}^{(1)^{-1}} \left[\mathbb{M}^{(1)} - \mathbb{E}^{2,1^T} \left(\mathbb{E}^{2,1} \mathbb{M}^{(1)^{-1}} \mathbb{E}^{2,1^T} \right)^{-1} \mathbb{E}^{2,1} \right] \mathbb{M}^{(1)^{-1}} \mathbb{N}_I^T,$$

$$\mathbb{F} = \mathbb{F}_{\hat{\varphi}} + \mathbb{F}_f,$$

$$\mathbb{F}_{\hat{\varphi}} = \mathbb{N}_I \mathbb{M}^{(1)^{-1}} \left[\mathbb{M}^{(1)} - \mathbb{E}^{2,1^T} \left(\mathbb{E}^{2,1} \mathbb{M}^{(1)^{-1}} \mathbb{E}^{2,1^T} \right)^{-1} \mathbb{E}^{2,1} \right] \mathbb{M}^{(1)^{-1}} \mathbb{N}_B^T \underline{\hat{\varphi}},$$

$$\mathbb{F}_f = -\mathbb{N}_I \mathbb{M}^{(1)^{-1}} \mathbb{E}^{2,1^T} \left(\mathbb{E}^{2,1} \mathbb{M}^{(1)^{-1}} \mathbb{E}^{2,1^T} \right)^{-1} \underline{f}.$$

- Inverting $\mathbb{M}^{(1)}$ and $\mathbb{E}^{2,1}\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^T}$ is easy (in parallel) because they are element-wise-block-diagonal.
- Solving for $\underline{\check{\varphi}}$ is cheap (smaller system size and condition number).
- Remaining local problems for $\underline{\boldsymbol{u}}$ and $\underline{\varphi}$ are trivial because $(\mathbb{E}^{2,1}\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^T})^{-1}$ is already computed.

Discretization

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

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We can easily eliminate $\underline{\boldsymbol{u}}$ and $\underline{\varphi}$ and obtain a system for the discrete interface variable $\underline{\check{\varphi}}$,

$$\mathbb{H}\underline{\check{\varphi}} = \mathbb{F},$$

where

$$\mathbb{H} = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - {\mathbb{E}^{2,1}}^T \left(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \right)^{-1} \mathbb{E}^{2,1} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_I^T,$$

$$\mathbb{F} = \mathbb{F}_{\hat{\varphi}} + \mathbb{F}_f,$$

$$\mathbb{F}_{\hat{\varphi}} = \mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - {\mathbb{E}^{2,1}}^T \left(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \right)^{-1} \mathbb{E}^{2,1} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_B^T \underline{\hat{\varphi}},$$

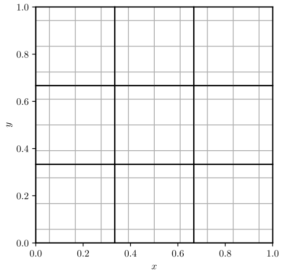
$$\mathbb{F}_f = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \left(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T \right)^{-1} \underline{f}.$$

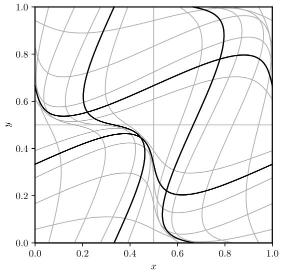
- Inverting $\mathbb{M}^{(1)}$ and $\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T$ is easy (in parallel) because they are element-wise-block-diagonal.
- Solving for $\underline{\check{\varphi}}$ is cheap (smaller system size and condition number).
- Remaining local problems for $\underline{\boldsymbol{u}}$ and $\underline{\varphi}$ are trivial because $(\mathbb{E}^{2,1} {\mathbb{M}^{(1)}}^{-1} {\mathbb{E}^{2,1}}^T)^{-1}$ is already computed.

Manufactured solution

Given a domain $\Omega = [0,1]^2$ and an exact solution $\varphi_{\text{exact}} = \cos(3\pi x e^y)$, we solve the Poisson problem with

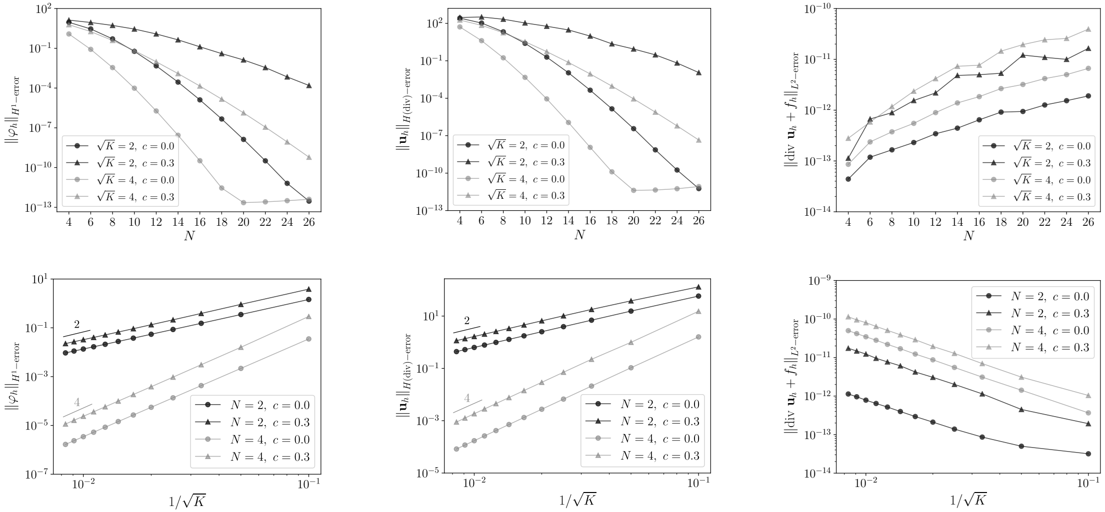
$$\begin{aligned} f_{\text{exact}} &= -\text{div grad}\, \varphi_{\text{exact}} && \text{in } \Omega, \\ \hat{\varphi} &= \text{tr}_{\text{grad}}\, \varphi_{\text{exact}} && \text{on } \partial\Omega. \end{aligned}$$



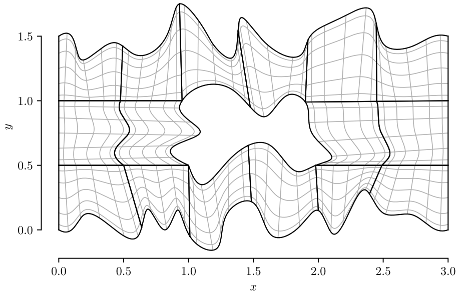


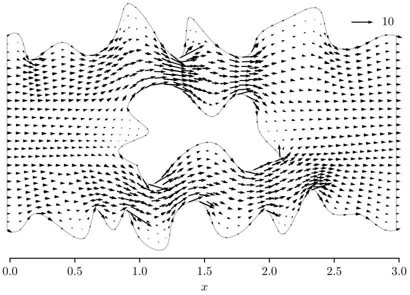
$$\begin{cases} x = \dfrac{1}{2} + \dfrac{1}{2}\left[\xi + c\sin(\pi\xi)\sin(\pi\eta)\right] \\ y = \dfrac{1}{2} + \dfrac{1}{2}\left[\eta + c\sin(\pi\xi)\sin(\pi\eta)\right] \end{cases}.$$

Manufactured solution



Potential flow in a domain with spline interpolation boundaries

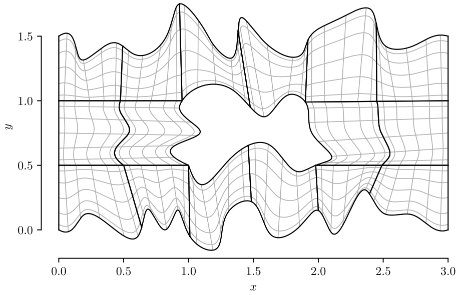


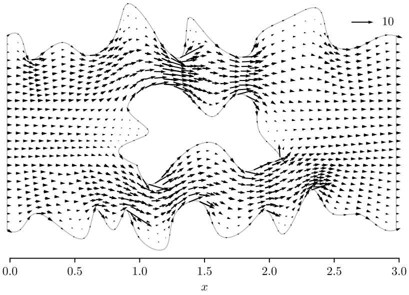


- Upper, lower and inner boundaries : Cubic splines interpolated free-slip walls.
- Left and right boundaries : Inlet and outlet of potential difference $\Delta\varphi = 10$.

Boundary	Sequence of samples.
Lower	$(0,0)$, $(0.11,0.01)$, $(0.20,0.12)$, $(0.61,-0.05)$, $(0.69,0.16)$, $(0.82,0)$, $(0.91,0.15)$, $(1.01,-0.05)$, $(1.21,-0.15)$, $(1.30,0.13)$, $(1.48,0.22)$, $(1.65,-0.05)$, $(1.85,0.02)$, $(2,0.15)$, $(2.11,-0.03)$, $(2.36,0.31)$, $(2.50,0.13)$, $(2.71,0.12)$, $(2.91,0)$, $(3,0)$.
Upper	$(0,1.5)$, $(0.09,1.51)$, $(0.17,1.32)$, $(0.43,1.45)$, $(0.58,1.36)$, $(0.83,1.50)$, $(0.93,1.75)$, $(1.14,1.52)$, $(1.18,1.45)$, $(1.33,1.33)$, $(1.4,1.64)$, $(1.59,1.45)$, $(1.88,1.37)$, $(1.92,1.47)$, $(2.15,1.63)$, $(2.40,1.71)$, $(2.51,1.43)$, $(2.72,1.42)$, $(2.89,1.5)$, $(3,1.5)$.
Inner	$(1,0.5)$, $(1.11,0.35)$, $(1.32,0.55)$, $(1.62,0.66)$, $(1.85,0.45)$, $(1.98,0.5)$, $(2.1,0.55)$, $(1.95,0.75)$, $(1.9,0.99)$, $(1.79,1.05)$, $(1.6,0.88)$, $(1.33,1.09)$, $(0.95,1)$, $(0.93,0.95)$, $(1.09,0.76)$, $(0.89,0.65)$, $(1,0.5)$.

Potential flow in a domain with spline interpolation boundaries





- Upper, lower and inner boundaries : Cubic splines interpolated free-slip walls.
- Left and right boundaries : Inlet and outlet of potential difference $\Delta\varphi = 10$.

TABLE – Fluxes through the domain.

N	Number of elements				
	16	64	256	576	1024
2	2.49949	2.92468	2.95905	3.01901	3.02207
4	2.95266	3.03115	3.02979	3.03123	3.03129
6	3.04810	3.02942	3.03120	3.03139	3.03139
8	3.01246	3.03047	3.03137	3.03140	3.03141
10	3.02062	3.03108	3.03141	3.03141	3.03141
12	3.03175	3.03137	3.03141	3.03141	3.03141
14	3.03045	3.03142	3.03141	3.03141	3.03141

Linear elasticity

Mixed formulation

Consider the Lagrange functional [10] for $(\underline{\sigma}, \underline{u}, \underline{\omega}) \in \underline{\underline{H}}(\text{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$:

$$\mathcal{L}(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}; \underline{f}, \underline{\hat{u}}) = \left(\underline{\underline{\sigma}}, \boldsymbol{C}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} - \left\langle \underline{\hat{u}}, \text{tr}_{\text{div}} \underline{\underline{\sigma}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega) \times \underline{\underline{H}}^{-1/2}(\partial\Omega)} + \left\langle \underline{u}, \text{div}\, \underline{\underline{\sigma}} + \underline{f} \right\rangle_{\underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)} - \left(\underline{\omega}, \boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)},$$

where $\underline{f} \in \underline{L}^2(\Omega)$ and $\underline{\hat{u}} = \text{tr}_{\text{grad}}\, \underline{u} \in \underline{H}^{1/2}(\partial\Omega)$ are given.

Variational analysis gives rise to following weak mixed formulation : *Find* $(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}) \in \underline{\underline{H}}(\text{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$ *such that*

$$\begin{cases} \left(\boldsymbol{C}\underline{\underline{\sigma}}, \underline{\underline{\check{\sigma}}}\right)_{L^2(\Omega)} + \left\langle \underline{u}, \text{div}\, \underline{\underline{\check{\sigma}}} \right\rangle_{\underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)} - \left(\underline{\omega}, \boldsymbol{T}\underline{\underline{\check{\sigma}}}\right)_{L^2(\Omega)} & = \left\langle \underline{\hat{u}}, \text{tr}_{\text{div}} \underline{\underline{\check{\sigma}}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega) \times \underline{\underline{H}}^{-1/2}(\partial\Omega)} \\ \left\langle \underline{\check{u}}, \text{div}\, \underline{\underline{\sigma}} \right\rangle_{\underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)} & = -\left\langle \underline{\check{u}}, \underline{f} \right\rangle_{\underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)} \\ -\left(\underline{\check{\omega}}, \boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} & = 0 \end{cases},$$

for all $(\underline{\underline{\check{\sigma}}}, \underline{\check{u}}, \underline{\check{\omega}}) \in \underline{\underline{H}}(\text{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$.

The solution of this weak formulation (the stationary point of the Lagrangian) solves the linear elasticity.

10. Olesen, K., Gervan, B., Reddy, J.N. and Gerritsma, M. A higher-order equilibrium finite element method, Int J Numer Methods Eng, (2018) 144 :1262-1290

Mixed formulation

Consider the Lagrange functional[10] for $(\underline{\sigma}, \underline{u}, \underline{\omega}) \in \underline{\underline{H}}(\text{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$:

$$\mathcal{L}(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}; \underline{f}, \underline{\hat{u}}) = \left(\underline{\underline{\sigma}}, \boldsymbol{C}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} - \left\langle \underline{\hat{u}}, \text{tr}_{\text{div}}\underline{\underline{\sigma}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega)} + \left\langle \underline{u}, \text{div}\ \underline{\underline{\sigma}} + \underline{f} \right\rangle_{\underline{\tilde{L}}^2(\Omega)\times\underline{L}^2(\Omega)} - \left(\underline{\omega}, \boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)},$$

where $\underline{f} \in \underline{L}^2(\Omega)$ and $\underline{\hat{u}} = \text{tr}_{\text{grad}}\ \underline{u} \in \underline{H}^{1/2}(\partial\Omega)$ are given.

Variational analysis gives rise to following weak mixed formulation : *Find* $(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}) \in \underline{\underline{H}}(\text{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$ *such that*

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$$\mathcal{L}(\underline{\sigma}, \underline{u}, \underline{\omega}; \underline{f}, \underline{\hat{u}}) = (\underline{\sigma}, \boldsymbol{C}\underline{\sigma})_{L^2(\Omega)} - \langle \underline{\hat{u}}, \mathrm{tr}_{\mathrm{div}} \underline{\sigma} \rangle_{\underline{H}^{1/2}(\partial\Omega) \times \underline{\underline{H}}^{-1/2}(\partial\Omega)} + \left\langle \underline{u}, \mathrm{div}\ \underline{\sigma} + \underline{f} \right\rangle_{\underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)} - (\underline{\omega}, \boldsymbol{T}\underline{\sigma})_{L^2(\Omega)},$$

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Hybrid mixed formulation

If we set up a mesh Ω^h in Ω, we get broken spaces :

$$\underline{\underline{H}}(\mathrm{div},\Omega^h),\ \underline{H}^1(\Omega^h),\ \underline{L}^2(\Omega^h).$$

Introduce a new Lagrange multiplier : $\underline{\bar{u}} \in \underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)$, we have a new functional :

$$\mathcal{L}(\underline{\underline{\sigma}},\underline{u},\underline{\omega},\underline{\bar{u}};\underline{f},\underline{\hat{u}}) = \left(\underline{\underline{\sigma}},\boldsymbol{C}\underline{\underline{\sigma}}\right)_{L^2(\Omega^h)} - \left\langle\underline{\hat{u}},\mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h)}$$
$$-\left\langle\underline{\bar{u}},\mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} + \left\langle\underline{u},\mathrm{div}\ \underline{\underline{\sigma}}+\underline{f}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \left(\underline{\omega},\boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega^h)},$$

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$$\begin{cases}
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\left\langle\underline{\check{u}},\mathrm{div}\ \underline{\underline{\sigma}}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} & = -\left\langle\underline{\check{u}},\underline{f}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\
-\left(\underline{\check{\omega}},\boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega^h)} & = 0 \\
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\end{cases},$$

for all $(\underline{\underline{\check{\sigma}}},\underline{\check{u}},\underline{\check{\omega}},\underline{\check{\bar{u}}}) \in \underline{\underline{H}}(\mathrm{div},\Omega^h)\times\underline{\tilde{L}}^2(\Omega^h)\times\underline{L}^2(\Omega^h)\times\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)$.

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If we set up a mesh Ω^h in Ω, we get broken spaces :

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$$\begin{cases} \left(\boldsymbol{C}\underline{\underline{\sigma}},\check{\underline{\underline{\sigma}}}\right)_{L^2(\Omega^h)} + \left\langle\underline{u},\mathrm{div}\ \check{\underline{\underline{\sigma}}}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} - \left(\underline{\omega},\boldsymbol{T}\check{\underline{\underline{\sigma}}}\right)_{L^2(\Omega^h)} - \left\langle\bar{\underline{u}},\mathrm{tr}_{\mathrm{div}}\check{\underline{\underline{\sigma}}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \left\langle\hat{\underline{u}},\mathrm{tr}_{\mathrm{div}}\check{\underline{\underline{\sigma}}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega)} \\ \left\langle\check{\underline{u}},\mathrm{div}\ \underline{\underline{\sigma}}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} & = -\left\langle\check{\underline{u}},\underline{f}\right\rangle_{\tilde{L}^2(\Omega^h)\times L^2(\Omega^h)} \\ -\left(\check{\underline{\omega}},\boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega^h)} & = 0 \\ -\left\langle\check{\bar{\underline{u}}},\mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

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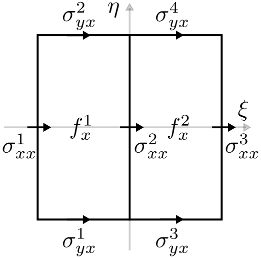
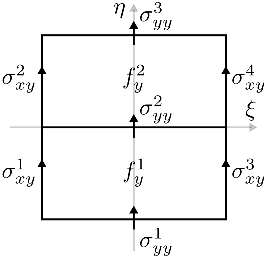
It is easy to prove that the interface variable $\bar{\underline{u}}$ represents the displacement on $\partial\Omega_h\backslash\partial\Omega$.

Discretization : Stress, body force and displacement





- For stress $\underline{\sigma}, \check{\underline{\sigma}}$; $\underline{H}(\mathrm{div},\Omega_k)$, we choose

$$\begin{bmatrix} \sigma_{xx}^h & \sigma_{yx}^h \\ \sigma_{xy}^h & \sigma_{yy}^h \end{bmatrix} \rightarrow \begin{bmatrix} \left\{h_i^{N+1}(\xi)e_j^{N-1}(\eta)\right\} & \left\{e_i^{N}(\xi)h_j^{N}(\eta)\right\} \\ \left\{h_i^{N}(\xi)e_j^{N}(\eta)\right\} & \left\{e_i^{N-1}(\xi)h_j^{N+1}(\eta)\right\} \end{bmatrix}.$$

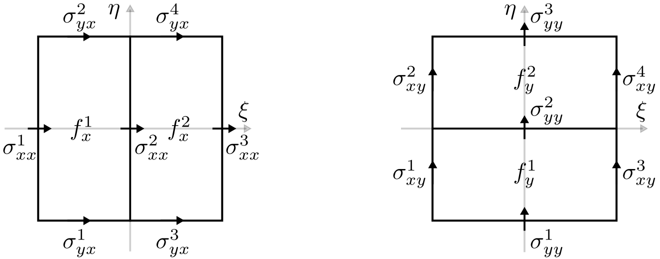
- For body force $\underline{f}$; $\underline{L}^2(\Omega_k)$, we choose

$$\left[f_x^h, f_y^h\right] \rightarrow \left[\left\{e_i^{N}(\xi)e_j^{N-1}(\eta)\right\}, \left\{e_i^{N-1}(\xi)e_j^{N}(\eta)\right\}\right].$$

- For displacement $\underline{u}, \check{\underline{u}}$; $\underline{\tilde{L}}^2(\Omega_k)$, we choose

$$\left[u_x^h, u_y^h\right] \rightarrow \left[\left\{\widetilde{e_i^{N}(\xi)e_j^{N-1}(\eta)}\right\}, \left\{\widetilde{e_i^{N-1}(\xi)e_j^{N}(\eta)}\right\}\right].$$

Discretization : Stress, body force and displacement



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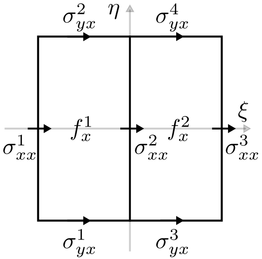
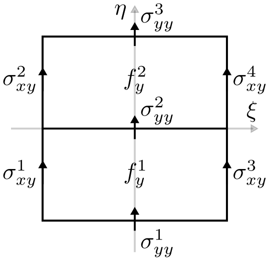
- For displacement $\underline{u}$, $\underline{\check{u}}$; $\underline{\tilde{L}}^2(\Omega_k)$, we choose

$$\left[u_x^h, u_y^h \right] \to \left[\left\{ \widetilde{e_i^N(\xi) e_j^{N-1}}(\eta) \right\}, \left\{ \widetilde{e_i^{N-1}(\xi) e_j^N}(\eta) \right\} \right].$$

Discretization : Stress, body force and displacement





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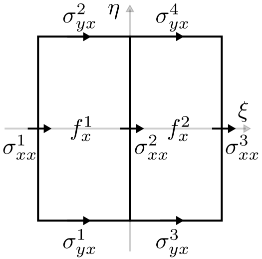
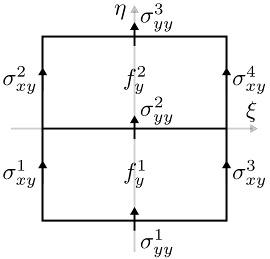
$$\left[f^h_x, f^h_y\right] \rightarrow \left[\left\{e_i^N(\xi)e_j^{N-1}(\eta)\right\}, \left\{e_i^{N-1}(\xi)e_j^N(\eta)\right\}\right].$$

- For displacement $\underline{u}$, $\underline{\check{u}}$; $\underline{\tilde{L}}^2(\Omega_k)$, we choose

$$\left[u^h_x, u^h_y\right] \rightarrow \left[\left\{e_i^N\widetilde{(\xi)e_j^{N-1}}(\eta)\right\}, \left\{e_i^{N-1}\widetilde{(\xi)e_j^N}(\eta)\right\}\right].$$

Discretization : Stress, body force and displacement





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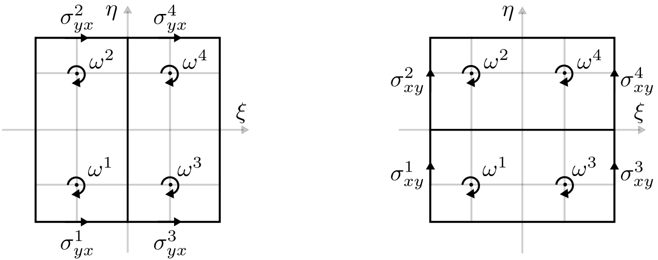
- For displacement $\underline{u}, \underline{\check{u}}; \underline{\tilde{L}}^2(\Omega_k)$, we choose

$$\left[u_x^h, u_y^h \right] \rightarrow \left[\left\{ e_i^N \widetilde{(\xi) e_j^{N-1}} (\eta) \right\}, \left\{ e_i^{N-1} \widetilde{(\xi) e_j^N} (\eta) \right\} \right].$$

Discretization : Rotation



- For rotation $\underline{\omega}, \underline{\check{\omega}}\,; \underline{L}^2(\Omega_k)$ which reduces to a scalar $\omega\,; L^2(\Omega_k)$ in $\mathbb{R}^2$, we choose

$$\omega \rightarrow \left\{ h_i^N(\xi) h_j^N(\eta) \right\}.$$

  It enforces the symmetry of the stress tensor in each element.
- In multiple element case, the kinematic spurious modes are there. So we have to loose the symmetry constraint by reduce the order of the polynomial by 1,

$$\omega \rightarrow \left\{ h_i^{N-1}(\xi) h_j^{N-1}(\eta) \right\}.$$

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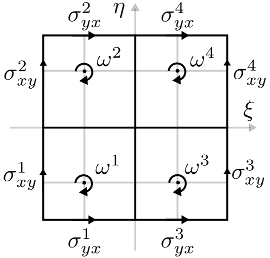

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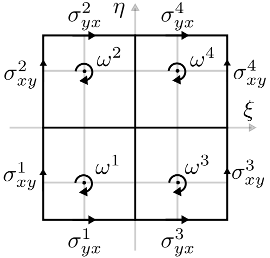

- For rotation $\underline{\omega}$, $\underline{\check{\omega}}$; $\underline{L}^2(\Omega_k)$ which reduces to a scalar ω ; $L^2(\Omega_k)$ in $\mathbb{R}^2$, we choose

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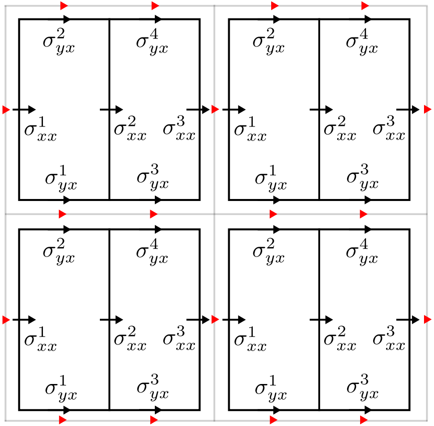
Discretization : Interface Lagrange multiplier

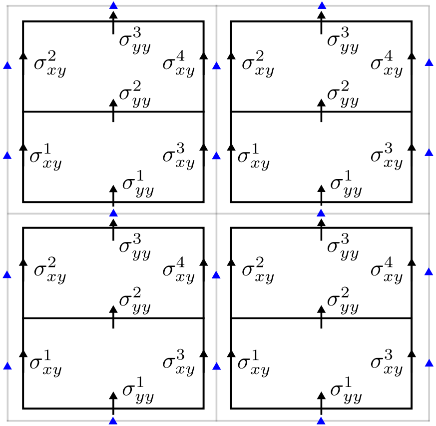


- For the Lagrange multiplier $\underline{\bar{u}}, \underline{\check{\bar{u}}}; \underline{H}^{1/2}(\partial\Omega_k)$, we choose

$$\bar{u}_x \rightarrow \left\{ \widetilde{e_i^N(\xi)}, \widetilde{e_i^N(\xi)}, \widetilde{e_i^{N-1}(\eta)}, \widetilde{e_i^{N-1}(\eta)} \right\},$$

corresponding to south, north, west and east boundaries of each element.

Discretization : Interface Lagrange multiplier



- For the Lagrange multiplier $\underline{\bar{u}}, \underline{\check{\bar{u}}}; \underline{H}^{1/2}(\partial\Omega_k)$, we choose

$$\bar{u}_y \rightarrow \left\{ \widetilde{e_i^{N-1}(\xi)}, \widetilde{e_i^{N-1}(\xi)}, \widetilde{e_i^{N}(\eta)}, \widetilde{e_i^{N}(\eta)} \right\},$$

corresponding to south, north, west and east boundaries of each element.

Discretization

Hybrid mixed formulation

Given $f \in \underline{L}^2(\Omega)$ *and* $\hat{\underline{u}} = \mathrm{tr}_{\mathrm{grad}}\, \underline{u} \in \underline{H}^{1/2}(\partial\Omega)$, *find* $(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}, \bar{\underline{u}}) \in \underline{\underline{H}}(\mathrm{div}, \Omega^h) \times \tilde{\underline{L}}^2(\Omega^h) \times \underline{L}^2(\Omega^h) \times \underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{C}\underline{\underline{\sigma}}, \check{\underline{\underline{\sigma}}})_{L^2(\Omega)} + \langle \underline{u}, \mathrm{div}\ \check{\underline{\underline{\sigma}}} \rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} - (\underline{\omega}, \boldsymbol{T}\check{\underline{\underline{\sigma}}})_{L^2(\Omega)} - \langle \bar{\underline{u}}, \mathrm{tr}_{\mathrm{div}} \check{\underline{\underline{\sigma}}} \rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times \underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \langle \hat{\underline{u}}, \mathrm{tr}_{\mathrm{div}} \check{\underline{\underline{\sigma}}} \rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega)} \\ \langle \check{\underline{u}}, \mathrm{div}\ \underline{\underline{\sigma}} \rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} & = -\left\langle \check{\underline{u}}, \underline{f} \right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} \\ -(\check{\underline{\omega}}, \boldsymbol{T}\underline{\underline{\sigma}})_{L^2(\Omega)} & = 0 \\ -\langle \check{\bar{\underline{u}}}, \mathrm{tr}_{\mathrm{div}} \underline{\underline{\sigma}} \rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times \underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

for all $(\check{\underline{\underline{\sigma}}}, \check{\underline{u}}, \check{\underline{\omega}}, \check{\bar{\underline{u}}}) \in \underline{\underline{H}}(\mathrm{div}, \Omega^h) \times \tilde{\underline{L}}^2(\Omega^h) \times \underline{L}^2(\Omega^h) \times \underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega)$.

Discrete hybrid mixed formulation is

$$\begin{bmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 & 0 \\ -\mathbb{T}^T & 0 & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \\ \boldsymbol{\omega} \\ \bar{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \hat{\boldsymbol{u}} \\ -\boldsymbol{f} \\ 0 \\ 0 \end{pmatrix}.$$

Discretization

Discrete hybrid mixed formulation :

$$\begin{bmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 & 0 \\ -\mathbb{T}^T & 0 & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \\ \boldsymbol{\omega} \\ \bar{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \hat{\boldsymbol{u}} \\ -\boldsymbol{f} \\ 0 \\ 0 \end{pmatrix}$$

- $\mathbb{M}^{(1)}$: element-wise-block-diagonal ; metric-dependent ;
- $\mathbb{T}$: element-wise-block-diagonal ; metric-dependent ;
- $\mathbb{E}^{2,1}$: element-wise block-diagonal ; metric-free ; ± 1 non-zero entries ; super sparse ;
- $\mathbb{N}$: metric-free ; ± 1 non-zero entries ; even more sparse ;

We can easily eliminate σ, u and ω, and obtain a system for the discrete interface variable $\bar{u}$,

$$\mathbb{H}\bar{u} = \mathbb{F},$$

where

$$\mathbb{H} = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - \mathbb{S}^T \left(\mathbb{S} {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \right)^{-1} \mathbb{S} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_I^T,$$

$$\mathbb{F} = \mathbb{F}_{\hat{u}} + \mathbb{F}_g,$$

$$\mathbb{F}_{\hat{u}} = \mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - \mathbb{S}^T \left(\mathbb{S} {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \right)^{-1} \mathbb{S} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_B^T \hat{u},$$

$$\mathbb{F}_g = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \left(\mathbb{S} {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \right)^{-1} g,$$

$$\mathbb{S}^T = \begin{bmatrix} {\mathbb{E}^{2,1}}^T & -\mathbb{T} \end{bmatrix}, \; g = \begin{pmatrix} -f^T & 0 \end{pmatrix}^T.$$

Discretization

Discrete hybrid mixed formulation :

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Discretization

Discrete hybrid mixed formulation :

$$\begin{bmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 & 0 \\ -\mathbb{T}^T & 0 & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \\ \boldsymbol{\omega} \\ \bar{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \hat{\boldsymbol{u}} \\ -\boldsymbol{f} \\ 0 \\ 0 \end{pmatrix}$$

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We can easily eliminate $\boldsymbol{\sigma}$, $\boldsymbol{u}$ and $\boldsymbol{\omega}$, and obtain a system for the discrete interface variable $\bar{\boldsymbol{u}}$,

$$\mathbb{H}\bar{\boldsymbol{u}} = \mathbb{F},$$

- Inverting $\mathbb{M}^{(1)}$ and $\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T$ is easy (in parallel) because they are element-wise-block-diagonal.
- Solving for $\bar{\boldsymbol{u}}$ is cheap (smaller system size and condition number).
- Remaining local problems for $\boldsymbol{\sigma}$, $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are trivial because $(\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T)^{-1}$ is already computed.

Manufactured solution

Given a domain $\Omega = [0,1]^2$, $E = 1$, $\nu = 0.3$ and exact solutions :

$$\boldsymbol{u} = \left[\sin(2\pi x)\cos(2\pi y),\ \cos(\pi x)\sin(\pi y)\right],\ \omega = -0.5\pi\sin(\pi x)\sin(\pi y) + \pi\sin(2\pi x)\sin(2\pi y),$$

$$\sigma_{xx} = \frac{E}{(1-v^2)}\left[2\pi\cos(2\pi x)\cos(2\pi y) + \nu\pi\cos(\pi x)\cos(\pi y)\right],\ \sigma_{yx} = \frac{E}{1+v}\left[-0.5\pi\sin(\pi x)\sin(\pi y) - \pi\sin(2\pi x)\sin(2\pi y)\right],$$

$$\sigma_{xy} = \frac{E}{1+v}\left[-0.5\pi\sin(\pi x)\sin(\pi y) - \pi\sin(2\pi x)\sin(2\pi y)\right],\ \sigma_{yy} = \frac{E}{(1-v^2)}\left[2\pi\nu\cos(2\pi x)\cos(2\pi y) + \pi\cos(\pi x)\cos(\pi y)\right],$$

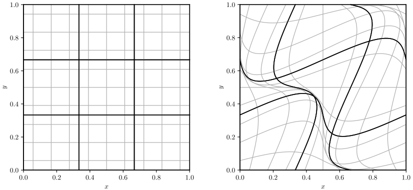
$$f_x = \frac{E}{(1-\nu^2)}\left[-4\pi^2\sin(2\pi x)\cos(2\pi y) - \nu\pi^2\sin(\pi x)\cos(\pi y)\right] + \frac{E}{1+\nu}\left[-0.5\pi^2\sin(\pi x)\cos(\pi y) - 2\pi^2\sin(2\pi x)\cos(2\pi y)\right],$$

$$f_y = \frac{E}{1+\nu}\left[-0.5\pi^2\cos(\pi x)\sin(\pi y) - 2\pi^2\cos(2\pi x)\sin(2\pi y)\right] + \frac{E}{(1-\nu^2)}\left[-4\pi^2\nu\cos(2\pi x)\sin(2\pi y) - \pi^2\cos(\pi x)\sin(\pi y)\right].$$

We solve the discrete hybrid mixed formulation in Ω with

$$f = f_{\text{exact}} \quad \text{in } \Omega,$$
$$\hat{u} = \text{tr}_{\text{grad}}\, u_{\text{exact}} \quad \text{on } \partial\Omega,$$

imposed in both orthogonal and heavily distorted meshes.



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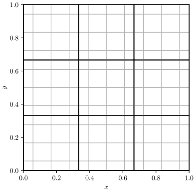
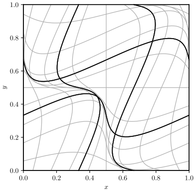
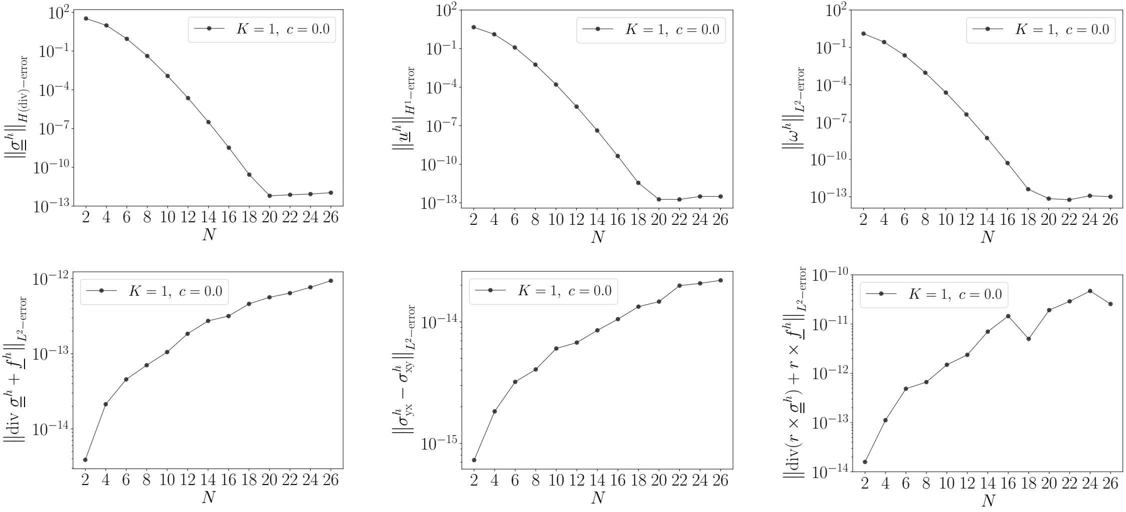




Manufactured solution : singular element



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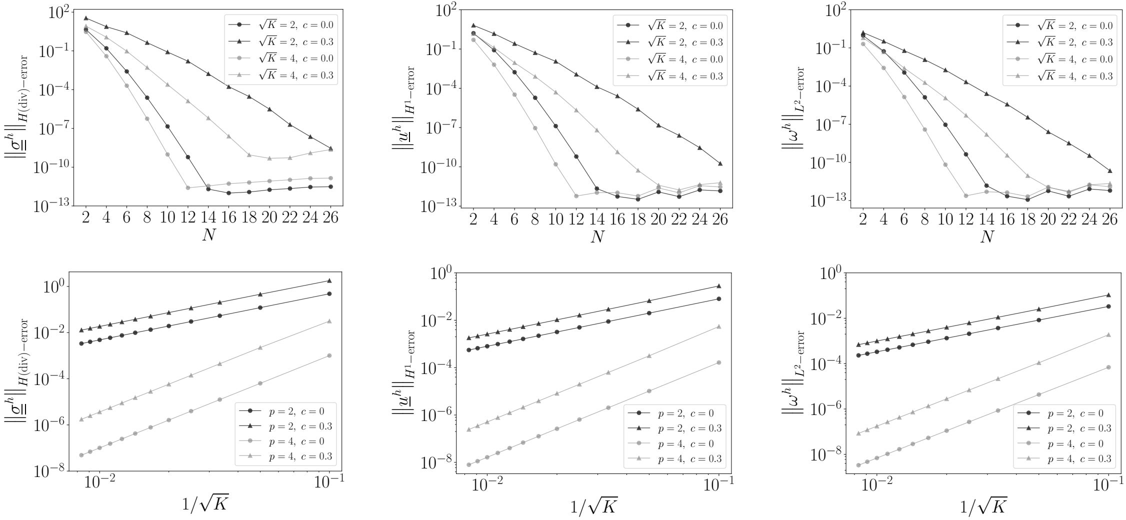

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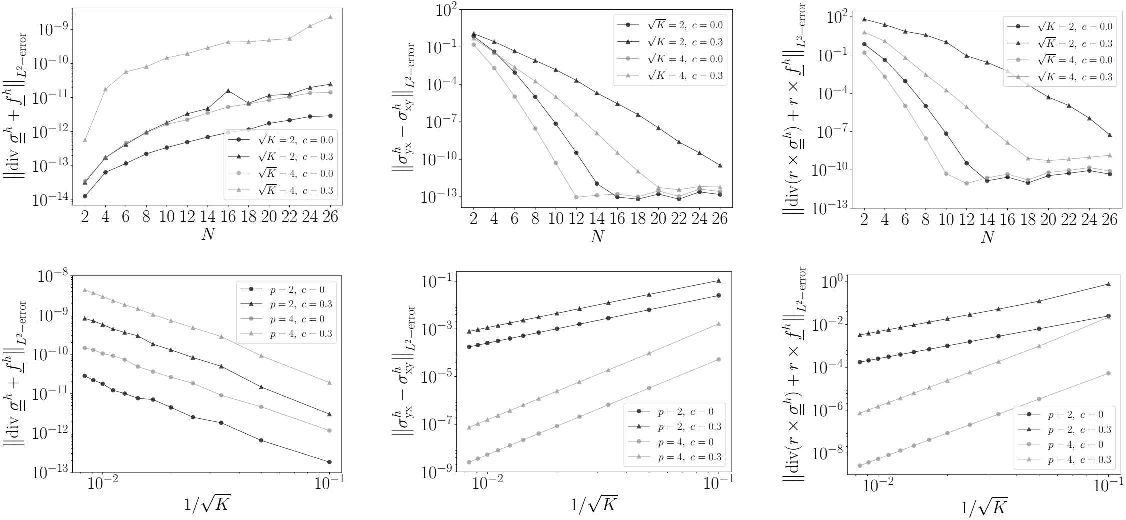


Crack : Opening

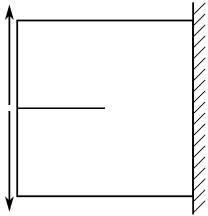


FIGURE – Opening crack.

- The geometry is $[-1,1]^2$ with a infinite crack at

$$x = [-1, 0],\ y = 0,$$

whose right side is mounted on a wall.

- Material properties : $E = 100$, $\nu = 0.3$.
- Opening shear stress :

$$\sigma_{xy}^{\text{up}} = 1,\ \sigma_{xy}^{\text{down}} = -1.$$

- Uniform *ph*-refinements.

Crack : In-plane shear

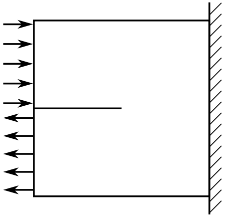


FIGURE – In plane shear crack.

- The geometry is $[-1,1]^2$ with a infinite crack at

$$x = [-1, 0],\ y = 0,$$

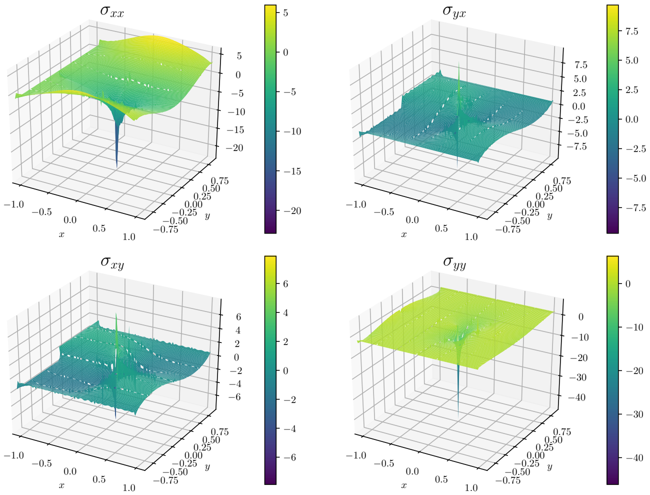
whose right side is mounted on a wall.

- Material properties : $E = 100$, $\nu = 0.3$.
- In plane shear normal stress :

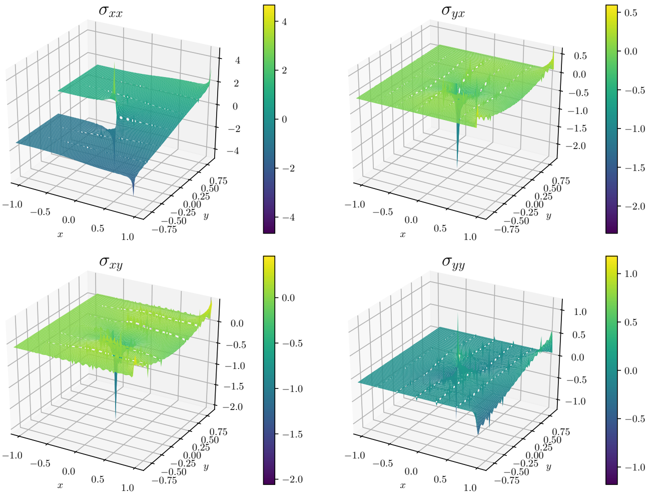
$$\sigma_{xx}^{\mathrm{up}} = 1,\ \sigma_{xx}^{\mathrm{down}} = -1.$$

- Uniform *ph*-refinements.

Crack : Opening, stress distribution.



Crack : In-plane shear, stress distribution.



Cracks : Complementary strain energy

TABLE – Opening.

N	number of elements 16	64	144	256	400	576
2	0.183928	0.180595	0.179565	0.179062	0.178764	0.178566
4	0.180120	0.178812	0.178399	0.178196	0.178075	0.177994
6	0.178970	0.178264	0.178038	0.177926	0.177860	0.177815
8	0.178468	0.178022	0.177878	0.177807	0.177764	0.177736
10	0.178202	0.177893	0.177792	0.177743	0.177713	0.177693
12	0.178043	0.177815	0.177741	0.177704	0.177682	0.177667
14	0.177940	0.177765	0.177708	0.177679	0.177662	0.177651

TABLE – In plane shear.

N	number of elements 16	64	144	256	400	576
2	0.0180946	0.0180009	0.0179741	0.0179613	0.0179538	0.0179488
4	0.0179924	0.0179557	0.0179450	0.0179398	0.0179368	0.0179348
6	0.0179619	0.0179421	0.0179362	0.0179333	0.0179317	0.0179306
8	0.0179486	0.0179361	0.0179323	0.0179305	0.0179294	0.0179287
10	0.0179415	0.0179328	0.0179302	0.0179289	0.0179282	0.0179277
12	0.0179373	0.0179309	0.0179289	0.0179280	0.0179274	0.0179270
14	0.0179346	0.0179296	0.0179281	0.0179274	0.0179269	0.0179266

Conclusions

We have proposed a high order spectral element method :

- The method uses integral values as dof's.
- The method is hybrid ; it is very easy to parallelize. Imposing boundary conditions is easy ; we have dof's on boundary for both Dirichlet and Neumann boundary conditions.
- The method is mimetic ; first-order differential operators can be preserved at the discrete level.
- The method uses dual polynomials ; some discrete matrices are metric-free, extremely sparse and low order finite-difference(volume)-like (containing non-zero entries of -1 and 1 only).
- It can be efficiently solved by solving a reduced system for the interface variable.

Further developments towards Stokes, Euler equations, Navier-Stokes are ongoing.

Thanks a lot. Questions ?

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- It can be efficiently solved by solving a reduced system for the interface variable.

Further developments towards Stokes, Euler equations, Navier-Stokes are ongoing.

Thanks a lot. Questions ?

Conclusions

We have propose a high order spectral element method :

- The method uses integral values as dof's.
- The method is hybrid ; it is very easy to parallelize. Imposing boundary conditions is easy ; we have dof's on boundary for both Dirichlet and Neumann boundary conditions.
- The method is mimetic ; first-order differential operators can be preserved at the discrete level.
- The method uses dual polynomials ; some discrete matrices are metric-free, extremely sparse and low order finite-difference(volume)-like (containing non-zero entries of -1 and 1 only).
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