A hybrid equilibrium formulation and the discretization with mimetic polynomials and their dual representations

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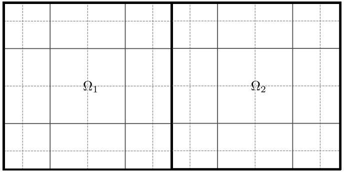
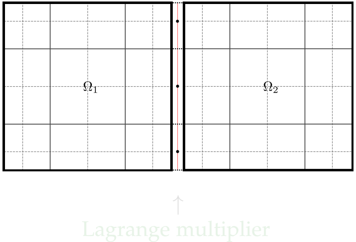
Summary

Hybrid methods

Hybrid (finite element) methods are those methods that relax the continuity across the inter-element interface by introducing a Lagrange multiplier between elements.





For more information about hybrid methods, we refer to Pian [1], Raviart and Thomas [2], Brezzi and Fortin [3].

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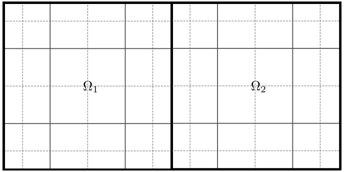

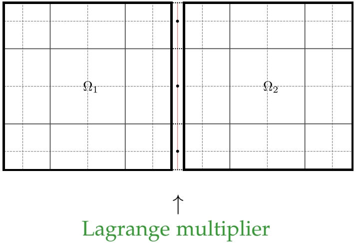

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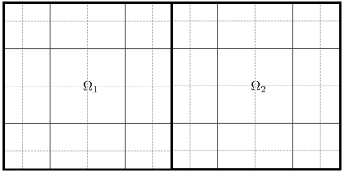

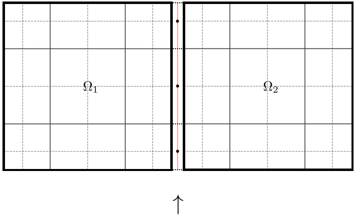

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Mimetic methods

Mimetic methods aim to preserve the structure of partial differential equations at the discrete level.

A key feature of mimetic mixed finite element methods is that their function spaces satisfy the De Rham complex :

$$\begin{array}{ccccccccc} \mathbb{R} \rightarrow & \Omega^{(0)} & \xrightarrow{\text{grad}} & \Omega^{(1)} & \xrightarrow{\text{curl}} & \Omega^{(2)} & \xrightarrow{\text{div}} & \Omega^{(3)} & \rightarrow 0, \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \mathbb{R} \rightarrow & \Omega_h^{(0)} & \xrightarrow{\text{grad}} & \Omega_h^{(1)} & \xrightarrow{\text{curl}} & \Omega_h^{(2)} & \xrightarrow{\text{div}} & \Omega_h^{(3)} & \rightarrow 0. \end{array}$$

Therefore, mimetic methods are also called structure-preserving methods.

Hybrid Mimetic Spectral Element Method [4, 5, 6] is a high order mimetic mixed finite element method.

4. Kreeft, J., Palha, A. and Gerritsma, M. Mimetic framework on curvilinear quadrilaterals of arbitrary order. *arXiv preprint*, (2011) arXiv :1111.4304.

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Preliminaries

Given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, let $L^2(\Omega)$ be the space of square integrable scalar-valued functions in Ω, then

$$H^1(\Omega) := \left\{ \varphi \in L^2(\Omega) \middle| \operatorname{grad} \varphi \in \left[L^2(\Omega)\right]^d \right\},$$

$$H(\mathrm{div},\Omega) := \left\{ \underline{u} \in \left[L^2(\Omega)\right]^d \middle| \operatorname{div} \underline{u} \in L^2(\Omega) \right\}.$$

And the trace spaces are defined as

$$H^{1/2}(\partial\Omega) := \mathrm{tr}_{\mathrm{grad}} H^1(\Omega), \quad H^{-1/2}(\partial\Omega) := \mathrm{tr}_{\mathrm{div}} H(\mathrm{div},\Omega),$$

which form a pair of dual spaces.

We further introduce notations :

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Mixed formulation

Re-write the Lagrange functional[7] for $(\underline{\sigma}, \underline{u}, \underline{\omega}) \in \underline{\underline{H}}(\mathrm{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$:

$$\mathcal{L}(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}; \underline{f}, \underline{\hat{u}}) = \left(\underline{\underline{\sigma}}, \boldsymbol{C}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} - \left\langle \underline{\hat{u}}, \mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{H}^{-1/2}(\partial\Omega)} + \left\langle \underline{u}, \mathrm{div}\ \underline{\underline{\sigma}} + \underline{f} \right\rangle_{\underline{\tilde{L}}^2(\Omega)\times\underline{L}^2(\Omega)} - \left(\underline{\omega}, \boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)},$$

where $\underline{f} \in \underline{L}^2(\Omega)$ and $\underline{\hat{u}} = \mathrm{tr}_{\mathrm{grad}}\ \underline{u} \in \underline{H}^{1/2}(\partial\Omega)$ are given.

Variational analysis gives rise to following weak mixed formulation : *Find* $(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}) \in \underline{\underline{H}}(\mathrm{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$ *such that*

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for all $(\underline{\underline{\check{\sigma}}}, \underline{\check{u}}, \underline{\check{\omega}}) \in \underline{\underline{H}}(\mathrm{div}, \Omega) \times \underline{\tilde{L}}^2(\Omega) \times \underline{L}^2(\Omega)$.

The solution of this weak formulation (the stationary point of the Lagrangian) solves the linear elasticity.

7. Olesen, K., Gervan, B., Reddy, J.N. and Gerritsma, M. A higher-order equilibrium finite element method, Int J Numer Methods Eng, (2018) 144 :1262-1290

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Broken Sobolev spaces

Given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. A mesh, denoted by Ω^h, partitions Ω into K disjoint open elements Ω_k with Lipschitz boundary $\partial\Omega_k$,

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k,\ \Omega_i \cap \Omega_j = \emptyset,\ 1 \leq i \neq j \leq K.$$

We can break $\underline{L}^2(\Omega), \underline{H}^1(\Omega), \underline{\underline{H}}(\mathrm{div},\Omega)$ and obtain the so-called broken Sobolev spaces [8] :

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Spaces for interface functions are then defined as

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8. Carstensen, C., Demkowicz, L. and Gopalakrishnan, J. Breaking spaces and forms for the DPG method and applications including Maxwell equations. *Computers and Mathematics with Applications*, (2016) 72(3) : 494-522.

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$$\underline{H}^1(\Omega^h) = \left\{ \underline{u} \in \underline{L}^2(\Omega) \middle| \underline{u}|_{\Omega_k} \in \underline{H}^1(\Omega_k) \right\} = \prod_{k=1}^{K} \underline{H}^1(\Omega_k),$$

$$\underline{\underline{H}}(\mathrm{div},\Omega^h) = \left\{ \underline{\underline{\sigma}} \in \left[\underline{L}^2(\Omega)\right]^d \middle| \underline{\underline{\sigma}}|_{\Omega_k} \in \underline{\underline{H}}(\mathrm{div},\Omega_k) \right\} = \prod_{k=1}^{K} \underline{\underline{H}}(\mathrm{div},\Omega_k).$$

Spaces for interface functions are then defined as

$$\underline{H}^{1/2}(\partial\Omega^h) := \mathrm{tr}^h_{\mathrm{grad}} \underline{H}^1(\Omega), \qquad \underline{\underline{H}}^{-1/2}(\partial\Omega^h) := \mathrm{tr}^h_{\mathrm{div}} \underline{\underline{H}}(\mathrm{div},\Omega),$$

which are a pair of dual spaces as well. $\mathrm{tr}^h_{\mathrm{grad}}$, $\mathrm{tr}^h_{\mathrm{div}}$ restrict $\underline{u} \in \underline{H}^1(\Omega)$, $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\mathrm{div},\Omega)$ onto $\partial\Omega_h = \bigcup_{k=1}^{K} \partial\Omega_k$.

8. Carstensen, C., Demkowicz, L. and Gopalakrishnan, J. Breaking spaces and forms for the DPG method and applications including Maxwell equations. *Computers and Mathematics with Applications*, (2016) 72(3) : 494-522.

Hybrid mixed formulation

If we set up a mesh Ω^h in Ω, we get broken spaces :

$$\underline{\underline{H}}(\mathrm{div},\Omega^h),\ \underline{H}^1(\Omega^h),\ \underline{L}^2(\Omega^h).$$

Introduce a new Lagrange multiplier : $\bar{\underline{u}} \in \underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)$, we have a new functional :

$$\mathcal{L}(\underline{\underline{\sigma}},\underline{u},\underline{\omega},\bar{\underline{u}};\underline{f},\hat{\underline{u}}) = \left(\underline{\underline{\sigma}},\boldsymbol{C}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} - \left\langle \hat{\underline{u}},\mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega)}$$
$$- \left\langle \bar{\underline{u}},\mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} + \left\langle \underline{u},\mathrm{div}\ \underline{\underline{\sigma}}+\underline{f}\right\rangle_{\tilde{\underline{L}}^2(\Omega)\times\underline{L}^2(\Omega)} - \left(\underline{\omega},\boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)},$$

Hybrid mixed formulation : *Given* $\underline{f} \in \underline{L}^2(\Omega)$ *and* $\hat{\underline{u}} = \mathrm{tr}_{\mathrm{grad}}\ \underline{u} \in \underline{H}^{1/2}(\partial\Omega)$, *find* $(\underline{\underline{\sigma}},\underline{u},\underline{\omega},\bar{\underline{u}}) \in \underline{\underline{H}}(\mathrm{div},\Omega^h) \times \tilde{\underline{L}}^2(\Omega^h)\times\underline{L}^2(\Omega^h)\times\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)$ *such that*

$$\begin{cases} \left(\boldsymbol{C}\underline{\underline{\sigma}},\check{\underline{\underline{\sigma}}}\right)_{L^2(\Omega)} + \left\langle \underline{u},\mathrm{div}\ \check{\underline{\underline{\sigma}}}\right\rangle_{\tilde{\underline{L}}^2(\Omega)\times\underline{L}^2(\Omega)} - \left(\underline{\omega},\boldsymbol{T}\check{\underline{\underline{\sigma}}}\right)_{L^2(\Omega)} - \left\langle \bar{\underline{u}},\mathrm{tr}_{\mathrm{div}}\check{\underline{\underline{\sigma}}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = \left\langle \hat{\underline{u}},\mathrm{tr}_{\mathrm{div}}\check{\underline{\underline{\sigma}}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega)} \\ \left\langle \check{\underline{u}},\mathrm{div}\ \underline{\underline{\sigma}}\right\rangle_{\tilde{\underline{L}}^2(\Omega)\times\underline{L}^2(\Omega)} & = -\left\langle \check{\underline{u}},\underline{f}\right\rangle_{\tilde{\underline{L}}^2(\Omega)\times\underline{L}^2(\Omega)} \\ -\left(\check{\underline{\omega}},\boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} & = 0 \\ -\left\langle \check{\bar{\underline{u}}},\mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}}\right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} & = 0 \end{cases},$$

for all $(\check{\underline{\underline{\sigma}}},\check{\underline{u}},\check{\underline{\omega}},\check{\bar{\underline{u}}}) \in \underline{\underline{H}}(\mathrm{div},\Omega^h)\times\tilde{\underline{L}}^2(\Omega^h)\times\underline{L}^2(\Omega^h)\times\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)$.

It is easy to prove that the interface variable $\bar{\underline{u}}$ represents the displacement on $\partial\Omega_h\backslash\partial\Omega$.

Hybrid mixed formulation

If we set up a mesh Ω^h in Ω, we get broken spaces :

$$\underline{\underline{H}}(\mathrm{div}, \Omega^h),\ \underline{H}^1(\Omega^h),\ \underline{L}^2(\Omega^h).$$

Introduce a new Lagrange multiplier : $\underline{\bar{u}} \in \underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega)$, we have a new functional :

$$\mathcal{L}(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}, \underline{\bar{u}}; \underline{f}, \underline{\hat{u}}) = \left(\underline{\underline{\sigma}}, \boldsymbol{C}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} - \left\langle \underline{\hat{u}}, \mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega)}$$

$$- \left\langle \underline{\bar{u}}, \mathrm{tr}_{\mathrm{div}}\underline{\underline{\sigma}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} + \left\langle \underline{u}, \mathrm{div}\ \underline{\underline{\sigma}} + \underline{f} \right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} - \left(\underline{\omega}, \boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)},$$

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Introduce a new Lagrange multiplier : $\bar{\underline{u}} \in \underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega)$, we have a new functional :

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$$- \langle \bar{\underline{u}}, \text{tr}_{\text{div}}\underline{\underline{\sigma}}\rangle_{\underline{H}^{1/2}(\partial\Omega^h\backslash\partial\Omega)\times\underline{\underline{H}}^{-1/2}(\partial\Omega^h\backslash\partial\Omega)} + \left\langle \underline{u}, \text{div}\ \underline{\underline{\sigma}} + \underline{f}\right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} - (\underline{\omega}, \boldsymbol{T}\underline{\underline{\sigma}})_{L^2(\Omega)},$$

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It is easy to prove that the interface variable $\bar{\underline{u}}$ represents the displacement on $\partial\Omega_h\backslash\partial\Omega$.

Mimetic basis functions

Let $-1=\xi_0<\xi_1<\cdots<\xi_N=1$ be a partitioning of the interval $[-1,1]$. The associated Lagrange polynomials :

$$h_i(\xi),\ \xi\in[-1,1],\ i=0,1,\cdots,N,\ \text{satisfying } h_i(\xi_j)=\delta_{i,j}\ \text{(Kronecker delta).}$$

The corresponding edge polynomials [9] are

$$e_i(\xi)=-\sum_{k=0}^{i-1}\frac{\mathrm{d}h_k(\xi)}{\mathrm{d}\xi}=\sum_{k=i}^{N}\frac{\mathrm{d}h_k(\xi)}{\mathrm{d}\xi},\ i=1,2,\cdots,N,\ \text{satisfying } \int_{\xi_{j-1}}^{\xi_j}e_i(\xi)=\delta_{i,j}.$$

9. Gerritsma, M. Edge functions for spectral element methods. *Spectral and High Order Methods for Partial Differential Equations.* Springer, (2011) 199-207

Mimetic basis functions

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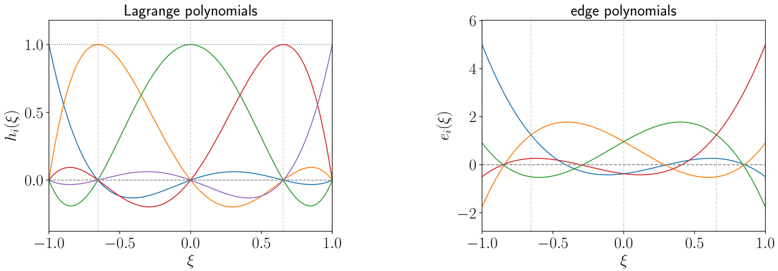

9. Gerritsma, M. Edge functions for spectral element methods. *Spectral and High Order Methods for Partial Differential Equations.* Springer, (2011) 199-207

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Finite dimensional spaces spanned by $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\}$ and $\{e_i(\xi)e_j(\eta)\}$ satisfy the De Rham complex. Let $\boldsymbol{u}, f$ be expanded as

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta)\right) \quad \text{and} \quad f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \text{div}\ \boldsymbol{u}$, then $f_h = \text{div}\ \boldsymbol{u}_h$ and

$$f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}\right) e_i(\xi) e_j(\eta) = \text{div}\ \boldsymbol{u}_h.$$

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Mimetic basis functions

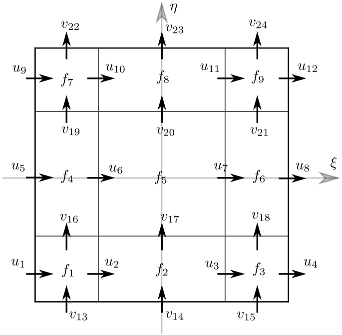

FIGURE – Reference domain.

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right),$$

$$f_h = \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \mathrm{div}\, \boldsymbol{u}$, then $f_h = \mathrm{div}\, \boldsymbol{u}_h$:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}\right) e_i(\xi) e_j(\eta).$$

Collect all equations and write them in vector form, we have

$$f = \mathbb{E}^{2,1} \boldsymbol{u},$$

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

$\mathbb{E}^{2,1}$ is the discrete div operator.

Mimetic basis functions

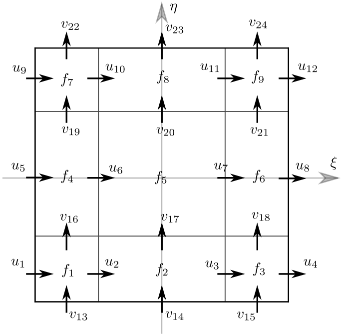

FIGURE – Reference domain.

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \right),$$

$$f_h = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta).$$

If $f = \text{div}\, \boldsymbol{u}$, then $f_h = \text{div}\, \boldsymbol{u}_h$:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} f_{i,j} e_i(\xi) e_j(\eta) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta).$$

Collect all equations and write them in vector form, we have

$$\boldsymbol{f} = \mathbb{E}^{2,1} \boldsymbol{u},$$

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

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Mimetic basis functions

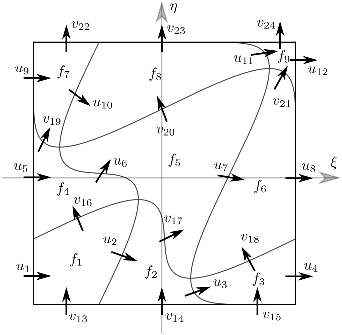


FIGURE – Curvilinear domain.

$$\boldsymbol{u}_h = \left(\sum_{i=0}^{N} \sum_{j=1}^{N} u'_{i,j}\; a_{i,j}(\xi,\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v'_{i,j}\; b_{i,j}(\xi,\eta) \right),$$

$$f_h = \sum_{i=1}^{N} \sum_{j=1}^{N} f'_{i,j}\; c_{i,j}(\xi,\eta).$$

If $f = \operatorname{div} \boldsymbol{u}$, then $f_h = \operatorname{div} \boldsymbol{u}_h$:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} f'_{i,j}\; c_{i,j}(\xi,\eta) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u'_{i,j} - u'_{i-1,j} + v'_{i,j} - v'_{i,j-1} \right) \; c_{i,j}(\xi,\eta).$$

Collect all equations and write them in vector form, we have

$$\boldsymbol{f}' = \mathbb{E}^{2,1} \boldsymbol{u}',$$

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

$\mathbb{E}^{2,1}$ is the discrete div operator.

Mimetic trace basis functions

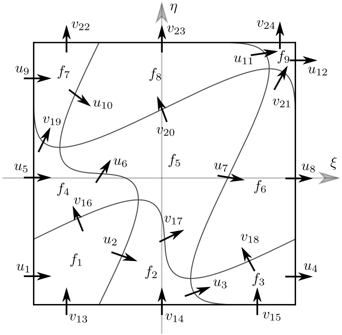


FIGURE – Curvilinear domain.

The trace variable $\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}$ can be discretized as

$$\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}_h = \left\{ \sum_{i=1}^{N} v_i^{\mathsf{s}} e_i'(\xi),\ \sum_{i=1}^{N} v_i^{\mathsf{n}} e_i'(\xi),\ \sum_{i=1}^{N} u_i^{\mathsf{w}} e_i'(\eta),\ \sum_{i=1}^{N} u_i^{\mathsf{e}} e_i'(\eta) \right\}.$$

There is a linear operator, $\mathbb{N}$, such that

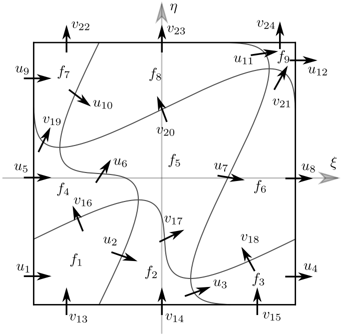
$$\boldsymbol{u}_{\mathrm{tr}}' = \mathbb{N}\boldsymbol{u}',$$

where $\boldsymbol{u}_{\mathrm{tr}}' = \left(-v_i^{\mathsf{s}}, v_i^{\mathsf{n}}, -u_i^{\mathsf{w}}, u_i^{\mathsf{e}}\right)^T$ and

$$\mathbb{N} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

$\mathbb{N}$ is the discrete trace operator.

Mimetic trace basis functions



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0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

$\mathbb{N}$ is the discrete trace operator.

Dual representations

Let scalar functions p_h and q_h both be expanded in terms of basis functions $\{e_i(\xi)e_j(\eta)\}$,

$$(p_h, q_h)_{L^2(\Omega)} = \boldsymbol{p}^T \mathbb{M}^{(2)} \boldsymbol{q},$$

where $\mathbb{M}^{(2)}$ is the mass matrix. We can further define the dual basis functions[10] as

$$\left[\widetilde{e_1(\xi)e_1(\eta)}, \cdots, \widetilde{e_N(\xi)e_N(\eta)}\right] := \left[e_1(\xi)e_1(\eta), \cdots, e_N(\xi)e_N(\eta)\right] {\mathbb{M}^{(2)}}^{-1}.$$

10. Jain, V., Zhang, Y., Palha, A. and Gerritsma, M. Construction and application of algebraic dual polynomial representations for finite element methods. (2017) arXiv :1712.09472.

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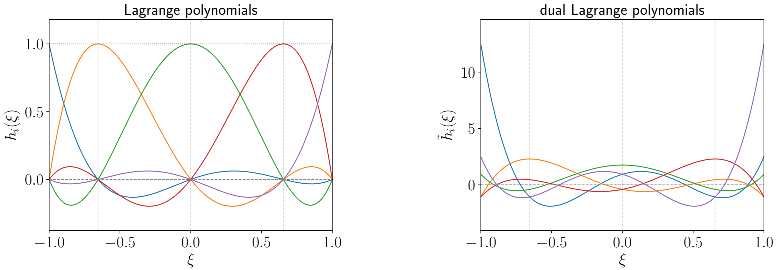

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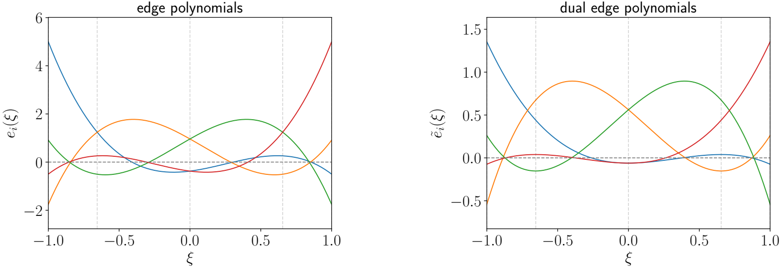

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If we expand $\tilde{p}_h$ in terms of the dual basis functions $\left\{ e_i\widetilde{(\xi)e_j}(\eta) \right\}$, we can obtain

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \tilde{\boldsymbol{p}}^T \boldsymbol{q}, \text{ where } \tilde{\boldsymbol{p}} = \mathbb{M}^{(2)} \boldsymbol{p},$$

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \langle \mathcal{M} p_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = (p_h, q_h)_{L^2(\Omega)}.$$

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Riesz Representation Theorem : *For every $\tilde{\boldsymbol{u}} \in \tilde{V}$, there exists a unique $\boldsymbol{u} \in V$, such that*

$$\langle \tilde{\boldsymbol{u}}, \boldsymbol{v} \rangle_{\tilde{V}\times V} = \langle \mathcal{R}\boldsymbol{u}, \boldsymbol{v} \rangle_{\tilde{V}\times V} = (\boldsymbol{u}, \boldsymbol{v})_V, \forall \boldsymbol{v} \in V,$$

$\mathcal{R} : \boldsymbol{u} \in V \rightarrow \tilde{\boldsymbol{u}} \in \tilde{V}$ *is called Riesz mapping*.

10. Jain, V., Zhang, Y., Palha, A. and Gerritsma, M. Construction and application of algebraic dual polynomial representations for finite element methods. (2017) arXiv :1712.09472.

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$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \tilde{\boldsymbol{p}}^T \boldsymbol{q}, \text{ where } \tilde{\boldsymbol{p}} = \mathbb{M}^{(2)} \boldsymbol{p},$$

$$\langle \tilde{p}_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \langle \mathcal{M} p_h, q_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = (p_h, q_h)_{L^2(\Omega)}.$$

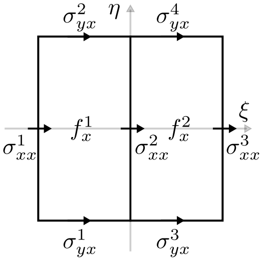
Furthermore, if $q_h = \text{div } \boldsymbol{v}_h$, and $\boldsymbol{v}_h$ is expanded by basis functions $\{h_i(\xi)e_j(\eta), e_i(\xi)h_j(\eta)\}$, we have

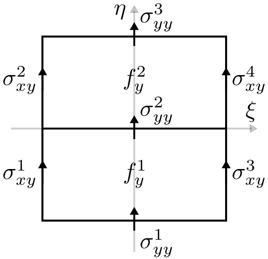
$$\langle \tilde{p}_h, \text{div } \boldsymbol{v}_h \rangle_{\tilde{L}(\Omega)\times L^2(\Omega)} = \tilde{\boldsymbol{p}}^T \mathbb{E}^{2,1} \boldsymbol{v}.$$

The same idea can be applied to the trace basis functions.

10. Jain, V., Zhang, Y., Palha, A. and Gerritsma, M. Construction and application of algebraic dual polynomial representations for finite element methods. (2017) arXiv :1712.09472.

Discretization : Stress, body force and displacement





- For stress $\underline{\sigma}$, $\underline{\check{\sigma}}$; $\underline{\underline{H}}(\mathrm{div},\Omega_k)$, we choose

$$\begin{bmatrix} \sigma_{xx}^h & \sigma_{yx}^h \\ \sigma_{xy}^h & \sigma_{yy}^h \end{bmatrix} \rightarrow \begin{bmatrix} \left\{ h_i^{N+1}(\xi) e_j^{N-1}(\eta) \right\} & \left\{ e_i^N(\xi) h_j^N(\eta) \right\} \\ \left\{ h_i^N(\xi) e_j^N(\eta) \right\} & \left\{ e_i^{N-1}(\xi) h_j^{N+1}(\eta) \right\} \end{bmatrix}.$$

- For body force $\underline{f}$; $\underline{L}^2(\Omega_k)$, we choose

$$\left[f_x^h, f_y^h \right] \rightarrow \left[\left\{ e_i^N(\xi) e_j^{N-1}(\eta) \right\}, \left\{ e_i^{N-1}(\xi) e_j^N(\eta) \right\} \right].$$

- For displacement $\underline{u}$, $\underline{\check{u}}$; $\underline{\tilde{L}}^2(\Omega_k)$, we choose

$$\left[u_x^h, u_y^h \right] \rightarrow \left[\left\{ \widetilde{e_i^N(\xi) e_j^{N-1}(\eta)} \right\}, \left\{ \widetilde{e_i^{N-1}(\xi) e_j^N(\eta)} \right\} \right].$$

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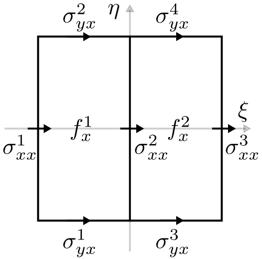

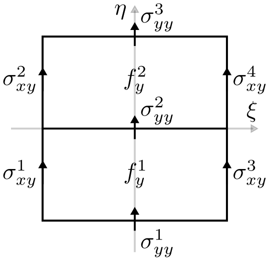

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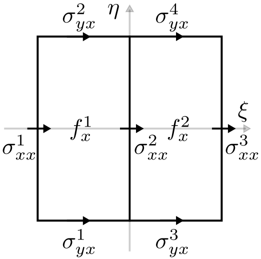

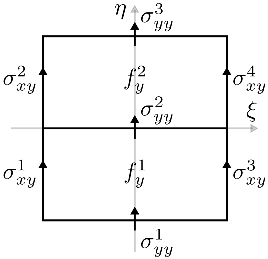

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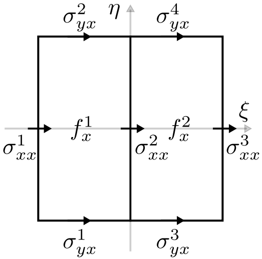

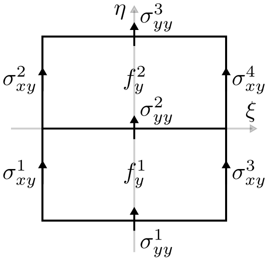

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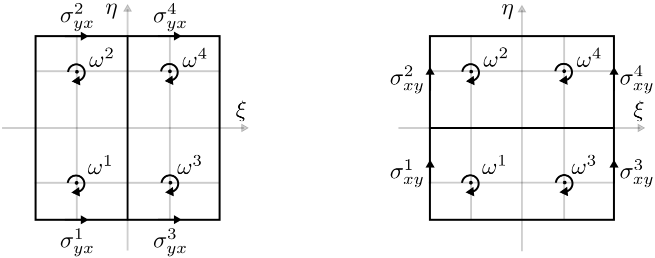
- For displacement $\underline{u}$, $\underline{\check{u}}$; $\underline{\tilde{L}}^2(\Omega_k)$, we choose

$$\left[u_x^h, u_y^h\right] \rightarrow \left[\left\{e_i^N\widetilde{(\xi)e_j^{N-1}}(\eta)\right\}, \left\{e_i^{N-1}\widetilde{(\xi)e_j^N}(\eta)\right\}\right].$$

Discretization : Rotation



- For rotation $\underline{\omega}, \underline{\check{\omega}}; \underline{L}^2(\Omega_k)$ which reduces to a scalar $\omega; L^2(\Omega_k)$ in $\mathbb{R}^2$, we choose

$$\omega \rightarrow \left\{ h_i^N(\xi) h_j^N(\eta) \right\}.$$

  It enforces the symmetry of the stress tensor in each element.

- In multiple element case, the kinematic spurious modes are there. So we have to loose the symmetry constraint by reduce the order of the polynomial by 1,

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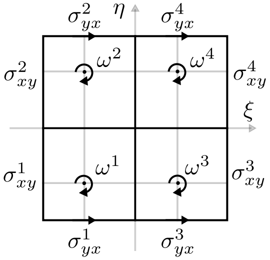
Discretization : Rotation



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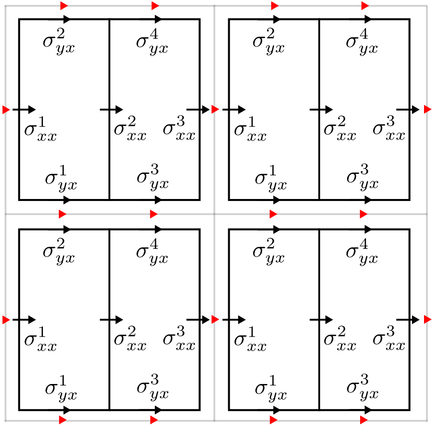
Discretization : Lagrange multiplier

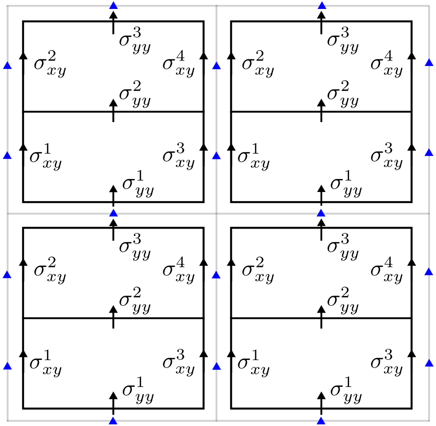


- For the Lagrange multiplier $\underline{\bar{u}}$, $\underline{\check{\bar{u}}}$; $\underline{H}^{1/2}(\partial\Omega_k)$, we choose

$$\bar{u}_x \rightarrow \left\{ \widetilde{e_i^N(\xi)}, \widetilde{e_i^N(\xi)}, \widetilde{e_i^{N-1}(\eta)}, \widetilde{e_i^{N-1}(\eta)} \right\},$$

corresponding to south, north, west and east boundaries of each element.

Discretization : Lagrange multiplier



- For the Lagrange multiplier $\underline{\bar{u}}$, $\underline{\check{\bar{u}}}$; $\underline{H}^{1/2}(\partial\Omega_k)$, we choose

$$\bar{u}_y \rightarrow \left\{ \wideparen{e_i^{N-1}(\xi)}, \wideparen{e_i^{N-1}(\xi)}, \wideparen{e_i^{N}(\eta)}, \wideparen{e_i^{N}(\eta)} \right\},$$

corresponding to south, north, west and east boundaries of each element.

Discretization

Hybrid mixed formulation

Given $f \in \underline{L}^2(\Omega)$ *and* $\hat{\underline{u}} = \mathrm{tr}_{\mathrm{grad}}\, \underline{u} \in \underline{H}^{1/2}(\partial\Omega)$, *find* $(\underline{\underline{\sigma}}, \underline{u}, \underline{\omega}, \bar{\underline{u}}) \in \underline{\underline{H}}(\mathrm{div}, \Omega^h) \times \tilde{\underline{L}}^2(\Omega^h) \times \underline{L}^2(\Omega^h) \times \underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega)$ *such that*

$$\begin{cases} \left(\boldsymbol{C}\underline{\underline{\sigma}}, \check{\underline{\underline{\sigma}}}\right)_{L^2(\Omega)} + \left\langle \underline{u}, \mathrm{div}\ \check{\underline{\underline{\sigma}}} \right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} - \left(\underline{\omega}, \boldsymbol{T}\check{\underline{\underline{\sigma}}}\right)_{L^2(\Omega)} - \left\langle \bar{\underline{u}}, \mathrm{tr}_{\mathrm{div}} \check{\underline{\underline{\sigma}}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega) \times \underline{\underline{H}}^{-1/2}(\partial\Omega^h \backslash \partial\Omega)} & = \left\langle \hat{\underline{u}}, \mathrm{tr}_{\mathrm{div}} \check{\underline{\underline{\sigma}}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega) \times \underline{\underline{H}}^{-1/2}(\partial\Omega)} \\ \left\langle \check{\underline{u}}, \mathrm{div}\ \underline{\underline{\sigma}} \right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} & = -\left\langle \check{\underline{u}}, \underline{f} \right\rangle_{\tilde{L}^2(\Omega)\times L^2(\Omega)} \\ -\left(\check{\underline{\omega}}, \boldsymbol{T}\underline{\underline{\sigma}}\right)_{L^2(\Omega)} & = 0 \\ -\left\langle \check{\bar{\underline{u}}}, \mathrm{tr}_{\mathrm{div}} \underline{\underline{\sigma}} \right\rangle_{\underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega) \times \underline{\underline{H}}^{-1/2}(\partial\Omega^h \backslash \partial\Omega)} & = 0 \end{cases},$$

for all $(\check{\underline{\underline{\sigma}}}, \check{\underline{u}}, \check{\underline{\omega}}, \check{\bar{\underline{u}}}) \in \underline{\underline{H}}(\mathrm{div}, \Omega^h) \times \tilde{\underline{L}}^2(\Omega^h) \times \underline{L}^2(\Omega^h) \times \underline{H}^{1/2}(\partial\Omega^h \backslash \partial\Omega)$.

Discrete hybrid mixed formulation is

$$\begin{bmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 & 0 \\ -\mathbb{T}^T & 0 & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \\ \boldsymbol{\omega} \\ \bar{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \hat{\boldsymbol{u}} \\ -\boldsymbol{f} \\ 0 \\ 0 \end{pmatrix}.$$

Discretization

Discrete hybrid mixed formulation :

$$\begin{bmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 & 0 \\ -\mathbb{T}^T & 0 & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \\ \boldsymbol{\omega} \\ \bar{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \hat{\boldsymbol{u}} \\ -\boldsymbol{f} \\ 0 \\ 0 \end{pmatrix}$$

- $\mathbb{M}^{(1)}$: element-wise-block-diagonal ; metric-dependent ;
- $\mathbb{T}$: element-wise-block-diagonal ; metric-dependent ;
- $\mathbb{E}^{2,1}$: element-wise block-diagonal ; metric-free ; ± 1 non-zero entries ; super sparse ;
- $\mathbb{N}$: metric-free ; ± 1 non-zero entries ; even more sparse ;

We can easily eliminate σ, u and ω, and obtain a system for the discrete interface variable $\bar{u}$,

$$\mathbb{H}\bar{u} = \mathbb{F},$$

where

$$\mathbb{H} = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - \mathbb{S}^T \left(\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T \right)^{-1} \mathbb{S} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_I^T,$$

$$\mathbb{F} = \mathbb{F}_{\hat{u}} + \mathbb{F}_g,$$

$$\mathbb{F}_{\hat{u}} = \mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \left[\mathbb{M}^{(1)} - \mathbb{S}^T \left(\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T \right)^{-1} \mathbb{S} \right] {\mathbb{M}^{(1)}}^{-1} \mathbb{N}_B^T \hat{\boldsymbol{u}},$$

$$\mathbb{F}_g = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \left(\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T \right)^{-1} g,$$

$$\mathbb{S}^T = \begin{bmatrix} {\mathbb{E}^{2,1}}^T & -\mathbb{T} \end{bmatrix}, \quad g = \begin{pmatrix} -f^T & 0 \end{pmatrix}^T.$$

Discretization

Discrete hybrid mixed formulation :

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$$\mathbb{F}_{\boldsymbol{g}} = -\mathbb{N}_I {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \left(\mathbb{S} {\mathbb{M}^{(1)}}^{-1} \mathbb{S}^T \right)^{-1} \boldsymbol{g},$$

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Discretization

Discrete hybrid mixed formulation :

$$\begin{bmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^T & -\mathbb{T} & -\mathbb{N}_I^T \\ \mathbb{E}^{2,1} & 0 & 0 & 0 \\ -\mathbb{T}^T & 0 & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \\ \boldsymbol{\omega} \\ \bar{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \hat{\boldsymbol{u}} \\ -\boldsymbol{f} \\ 0 \\ 0 \end{pmatrix}$$

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We can easily eliminate $\boldsymbol{\sigma}$, $\boldsymbol{u}$ and $\boldsymbol{\omega}$, and obtain a system for the discrete interface variable $\bar{\boldsymbol{u}}$,

$$\mathbb{H}\bar{\boldsymbol{u}} = \mathbb{F},$$

- Inverting $\mathbb{M}^{(1)}$ and $\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T$ is easy (in parallel) because they are element-wise-block-diagonal.
- Solving for $\bar{\boldsymbol{u}}$ is cheap (smaller system size and condition number).
- Remaining local problems for $\boldsymbol{\sigma}$, $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are trivial because $(\mathbb{S}{\mathbb{M}^{(1)}}^{-1}\mathbb{S}^T)^{-1}$ is already computed.

Manufactured solution

Given a domain $\Omega = [0,1]^2$, $E = 1$, $\nu = 0.3$ and exact solutions :

$$\boldsymbol{u} = [\sin(2\pi x)\cos(2\pi y),\ \cos(\pi x)\sin(\pi y)]\,,\ \omega = -0.5\pi\sin(\pi x)\sin(\pi y) + \pi\sin(2\pi x)\sin(2\pi y),$$

$$\sigma_{xx} = \frac{E}{(1-v^2)}\left[2\pi\cos(2\pi x)\cos(2\pi y) + \nu\pi\cos(\pi x)\cos(\pi y)\right],\ \sigma_{yx} = \frac{E}{1+v}\left[-0.5\pi\sin(\pi x)\sin(\pi y) - \pi\sin(2\pi x)\sin(2\pi y)\right],$$

$$\sigma_{xy} = \frac{E}{1+v}\left[-0.5\pi\sin(\pi x)\sin(\pi y) - \pi\sin(2\pi x)\sin(2\pi y)\right],\ \sigma_{yy} = \frac{E}{(1-v^2)}\left[2\pi v\cos(2\pi x)\cos(2\pi y) + \pi\cos(\pi x)\cos(\pi y)\right],$$

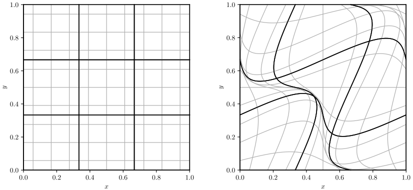
$$f_x = \frac{E}{(1-\nu^2)}\left[-4\pi^2\sin(2\pi x)\cos(2\pi y) - \nu\pi^2\sin(\pi x)\cos(\pi y)\right] + \frac{E}{1+\nu}\left[-0.5\pi^2\sin(\pi x)\cos(\pi y) - 2\pi^2\sin(2\pi x)\cos(2\pi y)\right],$$

$$f_y = \frac{E}{1+\nu}\left[-0.5\pi^2\cos(\pi x)\sin(\pi y) - 2\pi^2\cos(2\pi x)\sin(2\pi y)\right] + \frac{E}{(1-\nu^2)}\left[-4\pi^2\nu\cos(2\pi x)\sin(2\pi y) - \pi^2\cos(\pi x)\sin(\pi y)\right].$$

We solve the discrete hybrid mixed formulation in Ω with

$$\begin{aligned} f &= f_{\text{exact}} && \text{in } \Omega, \\ \hat{u} &= \text{tr}_{\text{grad}}\, u_{\text{exact}} && \text{on } \partial\Omega, \end{aligned}$$

imposed in both orthogonal and heavily distorted meshes.



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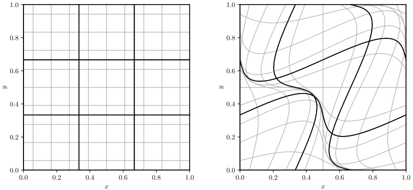
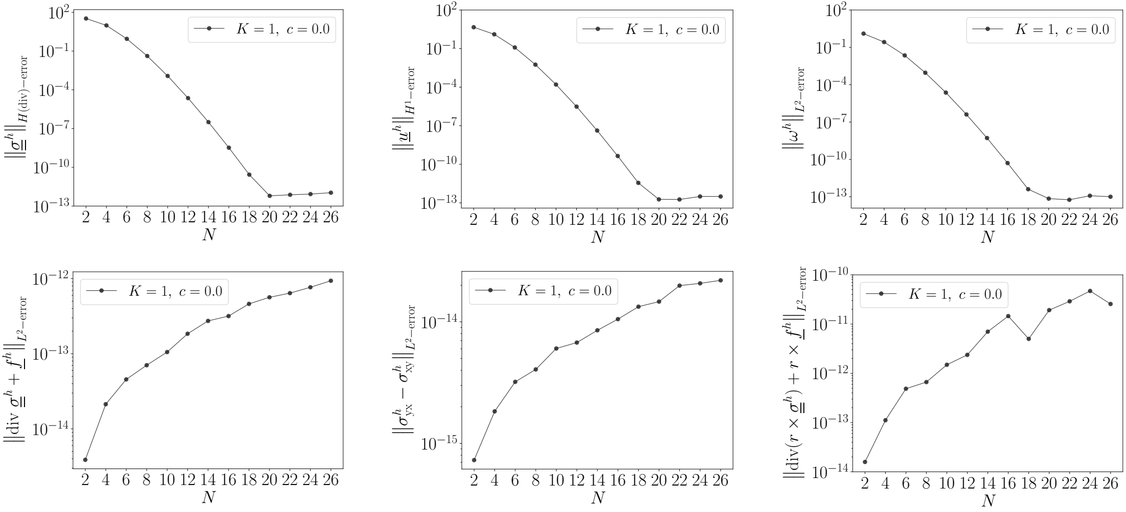
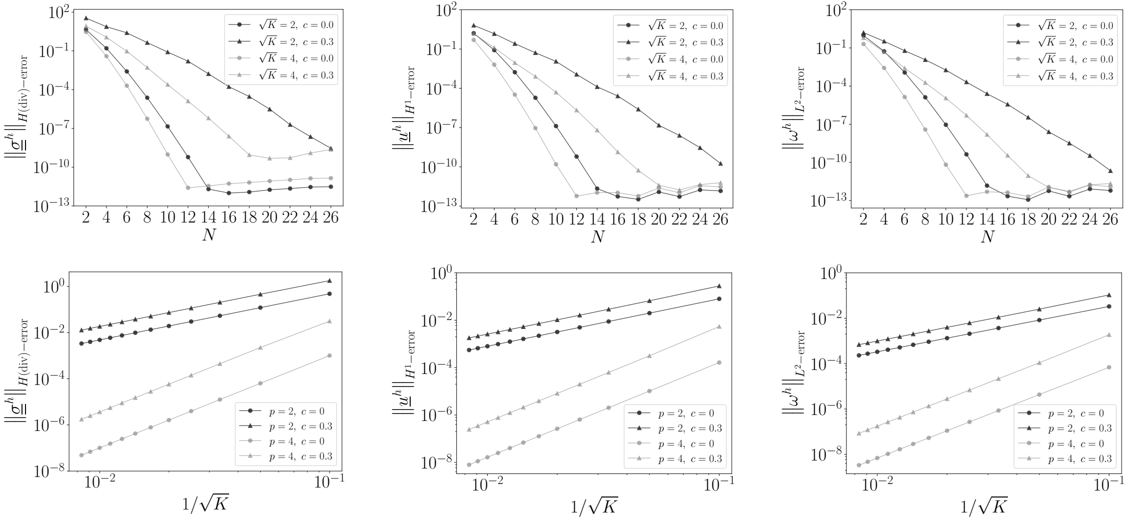


Manufactured solution : singular element



Manufactured solution



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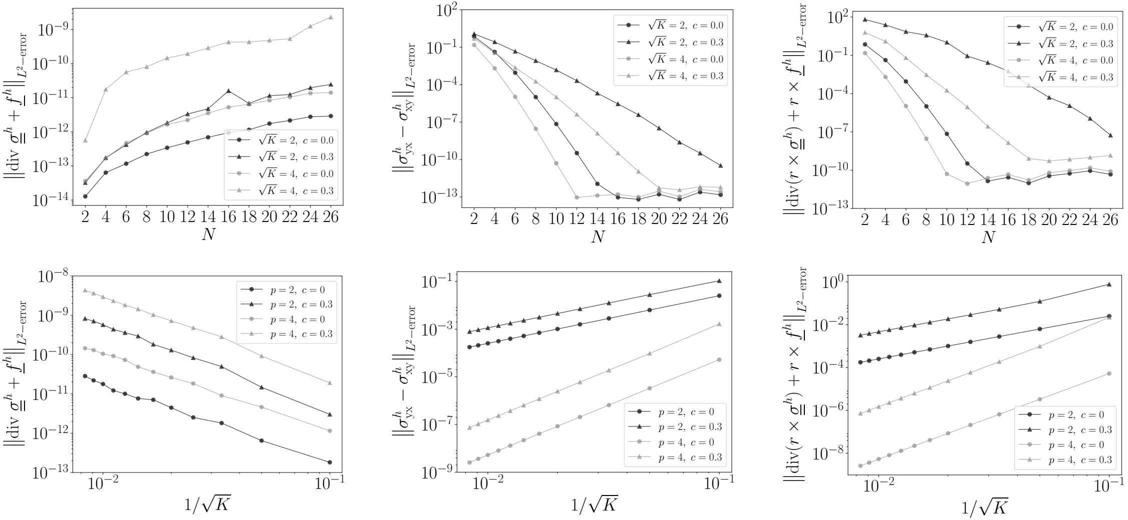


Crack : Opening

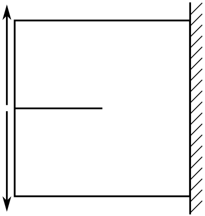


FIGURE – Opening crack.

- The geometry is $[-1,1]^2$ with a infinite crack at

$$x = [-1,0],\ y = 0,$$

whose right side is mounted on a wall.

- Material properties : $E = 100$, $\nu = 0.3$.
- Opening shear stress :

$$\sigma_{xy}^{\text{up}} = 1,\ \sigma_{xy}^{\text{down}} = -1.$$

- Uniformly *ph*-refinements.

Crack : In-plane shear

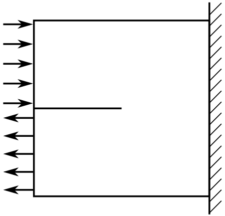


FIGURE – In plane shear crack.

- The geometry is $[-1,1]^2$ with a infinite crack at

$$x = [-1,0],\ y = 0,$$

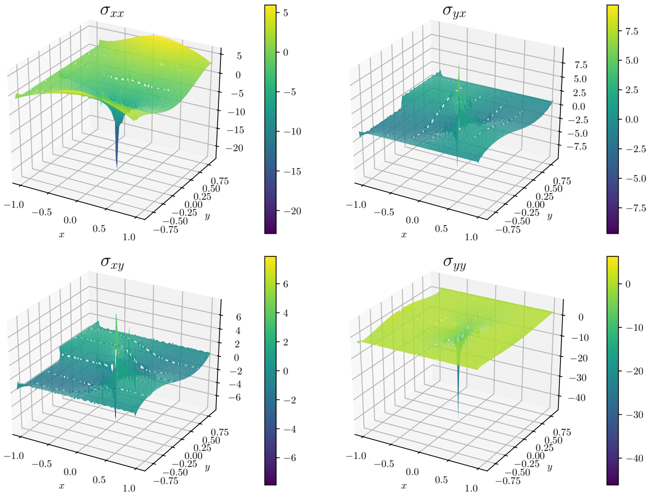
whose right side is mounted on a wall.

- Material properties : $E = 100$, $\nu = 0.3$.
- In plane shear normal stress :

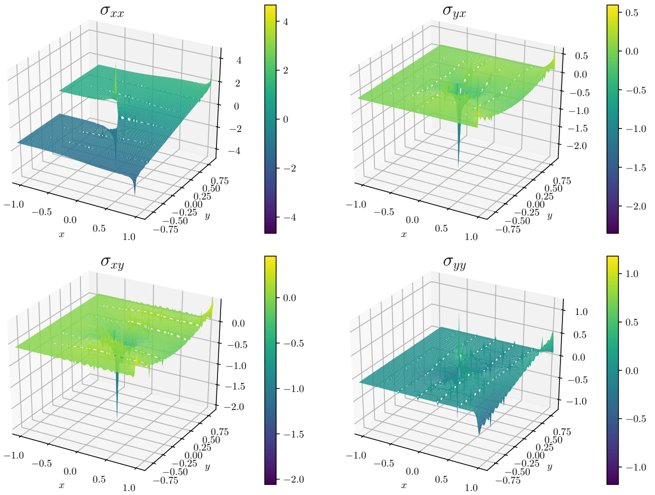
$$\sigma_{xx}^{\text{up}} = 1,\ \sigma_{xx}^{\text{down}} = -1.$$

- Uniformly *ph*-refinements.

Crack : Opening, stress distribution.



Crack : In-plane shear, stress distribution.



Cracks : Complementary strain energy

TABLE – Opening.

N	number of elements: 16	64	144	256	400	576
2	0.183928	0.180595	0.179565	0.179062	0.178764	0.178566
4	0.180120	0.178812	0.178399	0.178196	0.178075	0.177994
6	0.178970	0.178264	0.178038	0.177926	0.177860	0.177815
8	0.178468	0.178022	0.177878	0.177807	0.177764	0.177736
10	0.178202	0.177893	0.177792	0.177743	0.177713	0.177693
12	0.178043	0.177815	0.177741	0.177704	0.177682	0.177667
14	0.177940	0.177765	0.177708	0.177679	0.177662	0.177651

TABLE – In plane shear.

N	number of elements: 16	64	144	256	400	576
2	0.0180946	0.0180009	0.0179741	0.0179613	0.0179538	0.0179488
4	0.0179924	0.0179557	0.0179450	0.0179398	0.0179368	0.0179348
6	0.0179619	0.0179421	0.0179362	0.0179333	0.0179317	0.0179306
8	0.0179486	0.0179361	0.0179323	0.0179305	0.0179294	0.0179287
10	0.0179415	0.0179328	0.0179302	0.0179289	0.0179282	0.0179277
12	0.0179373	0.0179309	0.0179289	0.0179280	0.0179274	0.0179270
14	0.0179346	0.0179296	0.0179281	0.0179274	0.0179269	0.0179266

Conclusions

We have proposed a high order spectral element method for linear elasticity :

- The method uses integral values as dof's.
- The method is hybrid. So it is very easy to parallelize. And imposing boundary conditions is easy ; we have dof's on boundary for both Dirichlet and Neumann boundary conditions.
- The method is mimetic ; the divergence operator is preserved at the discrete level.
- The method uses dual polynomials. As a result, most blocks are metric-free, extremely sparse and low order finite-difference(volume)-like (containing non-zero entries of -1 and 1 only).
- It can be efficiently solved by solving a reduced system for the interface variable.

These features make the method a preferable one.

Thanks a lot. Questions ?

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