Mimetic Spectral Element Methods
With Hybridization and Algebraic Dual Basis Functions

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MS1001A: Domain Decomposition and Large-Scale Computation

Conservation in physics

In solid mechanics :

- Equilibrium of **forces**
- Equilibrium of **moments**

In fluid mechanics :

- Conservation of **mass**
- Conservation of **vorticity**
- Conservation of **kinetic energy**
- Conservation of **enstrophy**
- Conservation of **helicity**
- ......

Conservation laws are important.

Example : FVM in CFD.

*A fact : In most discretizations, conservations are only satisfied approximately. They will only be exactly satisfied at the limit of mesh refinement (**infinitely refined mesh**).*

discrete + **infinitely refined mesh** = continuous level

Mimetic discretizations : aim to satisfy/mimic structures of physics, like conservation laws, at the discrete level.

Structure-preserving discretizations, compatible discretizations.

Structure of PDE

We distinguish two types of basic relations in PDE : **Topological relations** and **Constitutive relations**.

- **Topological relations** : (de Rham complex)

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl};\Omega) \xrightarrow{\nabla\times} H(\mathrm{div};\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \longrightarrow 0,$$

- **Constitutive relations** : else ; usually related to the constitutive laws.

Example :

Poisson equation : $-\Delta\varphi = f$ or $-\nabla\cdot\nabla\varphi = f$, where $\varphi \in H^1(\Omega)$.

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$$\begin{cases} v = \nabla\varphi, & \text{so } v \in H(\mathrm{curl};\Omega) \\ \nabla\cdot v = -f, & \text{so } v \in H(\mathrm{div};\Omega) \end{cases}$$

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$$\begin{cases} \boldsymbol{v} = \nabla\varphi, & \\ \boldsymbol{u} = \star\boldsymbol{v}, & \boldsymbol{v} \in H(\mathrm{curl};\Omega) \overset{\star}{\leftrightarrow} \boldsymbol{u} \in H(\mathrm{div};\Omega) \\ \nabla\cdot\boldsymbol{u} = -f & \end{cases}$$

← *Topological relation*
← *Constitutive relation*
← *Topological relation;* (fluid : conservation of mass)

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$$\begin{cases} \boldsymbol{v} = \nabla\varphi, & \leftarrow \textit{Topological relation} \\ \boldsymbol{u} = \star\boldsymbol{v}, \quad \boldsymbol{v} \in H(\mathrm{curl};\Omega) \overset{\star}{\leftrightarrow} \boldsymbol{u} \in H(\mathrm{div};\Omega) & \leftarrow \textit{Constitutive relation} \\ \nabla\cdot\boldsymbol{u} = -f & \leftarrow \textit{Topological relation};\ \text{(fluid : conservation of mass)} \end{cases}$$

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Another example in Linear Elasticity :
constitutive relation $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$; *topological relation* $\nabla\cdot\boldsymbol{\sigma} = -\boldsymbol{f}$ (equilibrium of forces)

Mimetic spectral element method

With **Mimetic spectral element method (MSEM)** [1, 2, 3], we preserve the topological relations at the discrete level and allow approximation for *constitutive relation*.

Discrete de Rham complex :

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl};\Omega) \xrightarrow{\nabla\times} H(\mathrm{div};\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \longrightarrow 0,$$

$$\downarrow$$

$$\mathbb{R} \hookrightarrow H^1(\Omega_h) \xrightarrow{\nabla} H(\mathrm{curl};\Omega_h) \xrightarrow{\nabla\times} H(\mathrm{div};\Omega_h) \xrightarrow{\nabla\cdot} L^2(\Omega_h) \longrightarrow 0,$$

1. Kreeft, J., Palha, A. and Gerritsma, M. Mimetic framework on curvilinear quadrilaterals of arbitrary order. *arXiv preprint*, (2011) arXiv :1111.4304.
2. Kreeft, J. and Gerritsma, M. Mixed mimetic spectral element method for Stokes flow : A pointwise divergence-free solution. *Journal of Computational Physics*, (2013) 240 : 284-309.
3. Palha, A., Rebelo, P.P., Hiemstra, R., Kreeft, J. and Gerritsma, M. Physics-compatible discretization techniques on single and dual grids, with application to the Poisson equation of volume forms. *Journal of Computational Physics*, (2014) 257 : 1394-1422.

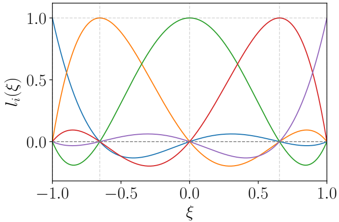
Mimetic spectral element method

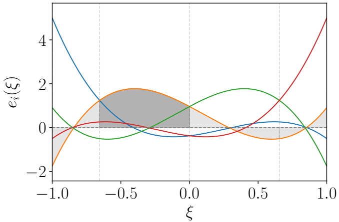
Let $-1 \leq \xi_0 < \xi_1 < \cdots < \xi_N \leq 1$. The well-known Lagrange polynomials are expressed by $l_i(\xi)$:

$$l_i(\xi) = \prod_{j=0, j\neq i}^{N} \frac{\xi - \xi_j}{\xi_i - \xi_j}, \quad i \in \{0, 1, 2, \cdots, N\}, \text{ satisfying, } l_i(\xi) = \delta_{i,j}.$$

The corresponding **edge polynomials** [4] are

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{\mathrm{d}l_k(\xi)}{\mathrm{d}\xi} = \sum_{k=i}^{N} \frac{\mathrm{d}l_k(\xi)}{\mathrm{d}\xi}, \; i \in \{1, 2, \cdots, N\} \text{ satisfying, } \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j}.$$





4. Gerritsma, M. Edge functions for spectral element methods. *Spectral and High Order Methods for Partial Differential Equations.* Springer, (2011) 199-207

Mimetic spectral element method

With tensor product, for example in $\mathbb{R}^3$,

$$\mathcal{P}:\{l_i(\xi)\otimes l_j(\eta)\otimes l_k(\varsigma)\}$$
$$\mathcal{E}:\{e_i(\xi)\otimes l_j(\eta)\otimes l_k(\varsigma),\quad l_i(\xi)\otimes e_j(\eta)\otimes l_k(\varsigma),\quad l_i(\xi)\otimes l_j(\eta)\otimes e_k(\varsigma)\}$$
$$\mathcal{F}:\{l_i(\xi)\otimes e_j(\eta)\otimes e_k(\varsigma),\quad e_i(\xi)\otimes l_j(\eta)\otimes e_k(\varsigma),\quad e_i(\xi)\otimes e_j(\eta)\otimes l_k(\varsigma)\}$$
$$\mathcal{V}:\{e_i(\xi)\otimes e_j(\eta)\otimes e_k(\varsigma)\}$$

Discrete de Rham complex : $$\mathbb{R}\hookrightarrow\mathcal{P}\xrightarrow{\nabla}\mathcal{E}\xrightarrow{\nabla\times}\mathcal{F}\xrightarrow{\nabla\cdot}\mathcal{V}\longrightarrow 0,$$

While in $\mathbb{R}^2$, finite dimensional spaces spanned by basis functions $\{h_i(\xi)e_j(\eta),e_i(\xi)h_j(\eta)\}$ and $\{e_i(\xi)e_j(\eta)\}$ satisfy the De Rham complex. Let vector-valued function $\boldsymbol{u}$ and scalar-valued function f be spanned into

$$\boldsymbol{u}_h=\left(\sum_{i=0}^{N}\sum_{j=1}^{N}u_{i,j}h_i(\xi)e_j(\eta),\sum_{i=1}^{N}\sum_{j=0}^{N}v_{i,j}e_i(\xi)h_j(\eta)\right)\quad\text{and}\quad f_h=\sum_{i=1}^{N}\sum_{j=1}^{N}f_{i,j}e_i(\xi)e_j(\eta).$$

If $f=\operatorname{div}\boldsymbol{u}$, then $f_h=\operatorname{div}\boldsymbol{u}_h$ and

$$f_h=\sum_{i=1}^{N}\sum_{j=1}^{N}\left(u_{i,j}-u_{i-1,j}+v_{i,j}-v_{i,j-1}\right)e_i(\xi)e_j(\eta),$$

Mimetic spectral element method

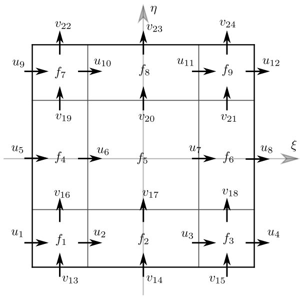


FIGURE – Reference domain.

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If $f = \operatorname{div} \boldsymbol{u}$, then

$$f_h = \operatorname{div} \boldsymbol{u}_h = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi) e_j(\eta).$$

Collect all equations and write them in vector form, we have

$$\underline{f} = \mathbb{E}^{2,1} \underline{u},$$

where

$$\mathbb{E}^{2,1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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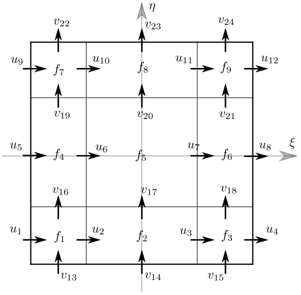

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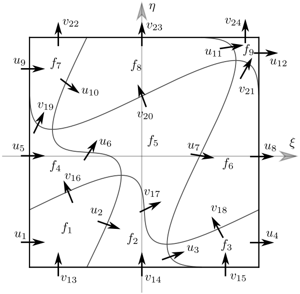

FIGURE – Curvilinear domain.

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Some literatures :

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Palha, A. and Gerritsma, M., A mass, energy, enstrophy and vorticity conserving (MEEVC) mimetic spectral element discretization for the 2D **incompressible Navier-Stokes equations**, *Journal of Computational Physics*, (2017) 328 : 200-220.

Lee D. and Palha A., A mixed mimetic spectral element model of the 3D **compressible Euler equations** on the cubed sphere, *Journal of Computational Physics*, (2019)

(In preparation), A mass, kinetic energy and helicity conserving mimetic spectral element method for **3D incompressible Navier-Stokes equations**.

Hybridization & dual basis functions

Mimetic spectral element method is computationally expensive.

- large systems
  - mixed formulation
  - high order method
- low sparsity

Not ready for large scale computation !

Hybridization & dual basis functions

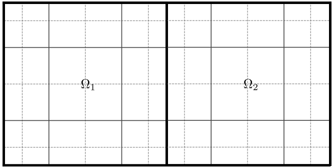
Mimetic spectral element method is computationally expensive.

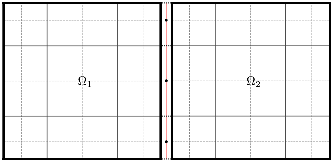
- large systems $\leftarrow$ Domain decomposition : Hybridization
  - mixed formulation
  - high order method
- low sparsity $\leftarrow$ Dual basis functions

Not ready for large scale computation !

Hybridization

Hybrid finite element methods are those methods that first allow the discontinuity across the inter-element interface then re-enforce (weakly or strongly) the continuity by introducing a Lagrange multiplier between elements.





↑
Lagrange multiplier

Similar idea has also be used in, e.g., *mortar methods* and *finite element tearing and interconnecting (FETI) methods*, and more.

Dual basis functions

Using dual basis functions eliminates some metric-dependent matrices from the discrete system.

$$(\phi, \varphi) = \underline{\phi}^{\mathsf{T}} \mathbb{M} \underline{\varphi}$$

$$(\phi, \widetilde{\varphi}) = \underline{\phi}^{\mathsf{T}} \underline{\widetilde{\varphi}}$$

where $\widetilde{\varphi}$ represent it is expanded with dual basis functions.

Example : Poisson equation [5]

The well-known mixed formulation of the Poisson equation : *Find* $(\boldsymbol{u}, \varphi) \in H(\text{div}, \Omega) \times H^1(\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v}) + (\varphi, \text{div } \boldsymbol{v}) & = \langle \hat{\varphi}, \text{tr}_{\text{div}} \boldsymbol{v} \rangle_{\partial\Omega} \\ (\psi, \text{div } \boldsymbol{u}) & = -(\psi, f) \end{cases},$$

for all $(\boldsymbol{v}, \psi) \in H(\text{div}, \Omega) \times H^1(\Omega)$. This is a weak mixed formulation of the Poisson equation.

Discretization :

$$\begin{pmatrix} \mathbb{M}_1 & (\mathbb{M}_2 \mathbb{E})^T \\ \mathbb{M}_2 \mathbb{E} & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \end{pmatrix} = \begin{pmatrix} \mathbb{B} \\ -\mathbb{M}_2 \underline{f} \end{pmatrix}.$$

5. Y. Zhang, V. Jain, A. Palha, M. Gerritsma, The discrete Steklov-Poincare operator using algebraic dual polynomials, Computational Methods in Applied Mathematics, (2019)

Example : Poisson equation, hybridization

If we set up a mesh Ω_h in Ω, by breaking $\boldsymbol{u}$ and φ into broken spaces, $H(\mathrm{div},\Omega_h)$ and $H^1(\Omega_h)$, and introducing a new Lagrange multiplier $\check{\varphi}$ in the interface space $H^{1/2}(\partial\Omega_h\backslash\partial\Omega)$

Given $f\in L^2(\Omega)$ *and* $\hat{\varphi}=\mathrm{tr}_{\mathrm{grad}}\,\varphi\in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u},\varphi,\check{\varphi})\in H(\mathrm{div},\Omega_h)\times H^1(\Omega_h)\times H^{1/2}(\partial\Omega_h\backslash\partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u},\boldsymbol{v})+(\varphi,\mathrm{div}\,\boldsymbol{v})-\langle\check{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\rangle_{\partial\Omega_h\backslash\partial\Omega} & =\langle\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\rangle \\ (\psi,\mathrm{div}\,\boldsymbol{u}) & =-(\psi,f) \\ -\langle\check{\psi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\rangle_{\partial\Omega_h\backslash\partial\Omega} & =0 \end{cases},$$

for all $(\boldsymbol{v},\psi,\check{\psi})\in H(\mathrm{div},\Omega_h)\times H^1(\Omega_h)\times H^{1/2}(\partial\Omega_h\backslash\partial\Omega)$.

Discretization :

$$\begin{pmatrix} \mathbb{M}_1 & \mathbb{M}_2\mathbb{E}^T & -(\mathbb{MN}_I)^T \\ \mathbb{M}_2\mathbb{E} & 0 & 0 \\ -\mathbb{MN}_I & 0 & 0 \end{pmatrix}\begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix}=\begin{pmatrix} \mathbb{B}\underline{\hat{\varphi}} \\ -\mathbb{M}_2\underline{f} \\ 0 \end{pmatrix}.$$

Example : Poisson equation, hybridization + dual basis functions

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Given $f\in L^2(\Omega)$ *and* $\hat{\varphi}=\mathrm{tr}_{\mathrm{grad}}\ \varphi\in H^{1/2}(\partial\Omega)$, *find* $(\boldsymbol{u},\varphi,\check{\varphi})\in H(\mathrm{div},\Omega_h)\times H^1(\Omega_h)\times H^{1/2}(\partial\Omega_h\backslash\partial\Omega)$ *such that*

$$\begin{cases} (\boldsymbol{u},\boldsymbol{v})+(\varphi,\mathrm{div}\ \boldsymbol{v})-\langle\check{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\rangle_{\partial\Omega_h\backslash\partial\Omega} & =\langle\hat{\varphi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{v}\rangle \\ (\psi,\mathrm{div}\ \boldsymbol{u}) & =-(\psi,f) \\ -\langle\check{\psi},\mathrm{tr}_{\mathrm{div}}\boldsymbol{u}\rangle_{\partial\Omega_h\backslash\partial\Omega} & =0 \end{cases},$$

for all $(\boldsymbol{v},\psi,\check{\psi})\in H(\mathrm{div},\Omega_h)\times H^1(\Omega_h)\times H^{1/2}(\partial\Omega_h\backslash\partial\Omega)$.

Discretization :

$$\begin{pmatrix}\mathbb{M}_1 & \mathbb{E}^T & -\mathbb{N}_I^T\\ \mathbb{E} & 0 & 0\\ -\mathbb{N}_I & 0 & 0\end{pmatrix}\begin{pmatrix}\underline{\boldsymbol{u}}\\ \underline{\varphi}\\ \underline{\check{\varphi}}\end{pmatrix}=\begin{pmatrix}\mathbb{N}_B^T\underline{\hat{\varphi}}\\ -\underline{f}\\ 0\end{pmatrix}.$$

Example : Poisson equation, hybridization + dual basis functions

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}_1 & \mathbb{E}^T & -\mathbb{N}_I^T \\ \mathbb{E} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

- $\mathbb{M}_1$: metric-dependent ; element-wise-block-diagonal ;
- $\mathbb{E}$: metric-independent ; element-wise-block-diagonal ; super sparse ; ± 1 non-zero entries ;
- $\mathbb{N}$: metric-independent ; even more sparse ; ± 1 non-zero entries ;

$$\mathbb{E} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.$$

Example : Poisson equation, hybridization + dual basis functions

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}_1 & \mathbb{E}^T & -\mathbb{N}_I^T \\ \mathbb{E} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

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- $\mathbb{N}$: metric-independent ; even more sparse ; ± 1 non-zero entries ;

$$\mathbb{N} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

Example : Poisson equation, hybridization + dual basis functions

Discrete hybrid mixed formulation :

$$\begin{pmatrix} \mathbb{M}_1 & \mathbb{E}^T & -\mathbb{N}_I^T \\ \mathbb{E} & 0 & 0 \\ -\mathbb{N}_I & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\varphi} \\ \underline{\check{\varphi}} \end{pmatrix} = \begin{pmatrix} \mathbb{N}_B^T \underline{\hat{\varphi}} \\ -\underline{f} \\ 0 \end{pmatrix}.$$

- $\mathbb{M}_1$: metric-dependent ; element-wise-block-diagonal ;
- $\mathbb{E}$: metric-independent ; element-wise-block-diagonal ; super sparse ; ± 1 non-zero entries ;
- $\mathbb{N}$: metric-independent ; even more sparse ; ± 1 non-zero entries ;

A reduced system for the interface variable $\underline{\check{\varphi}}$:

$$\mathbb{S}\underline{\check{\varphi}} = \mathsf{F},$$

where

$$\mathbb{S} = -\mathbb{N}_I {\mathbb{M}_1}^{-1} \left[\mathbb{M}_1 - \mathbb{E}^T \left(\mathbb{E}{\mathbb{M}_1}^{-1}\mathbb{E}^T \right)^{-1} \mathbb{E} \right] \mathbb{M}_1^{-1} \mathbb{N}_I^T,$$

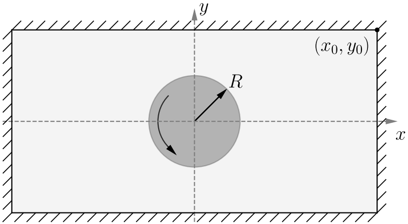
$$\mathsf{F} = \mathsf{F}_{\hat{\varphi}} + \mathsf{F}_f,$$

$$\mathsf{F}_{\hat{\varphi}} = \mathbb{N}_I {\mathbb{M}_1}^{-1} \left[\mathbb{M}_1 - \mathbb{E}^T \left(\mathbb{E}{\mathbb{M}_1}^{-1}\mathbb{E}^T \right)^{-1} \mathbb{E} \right] {\mathbb{M}_1}^{-1} \mathbb{N}_B^T \underline{\hat{\varphi}},$$

$$\mathsf{F}_f = -\mathbb{N}_I {\mathbb{M}_1}^{-1} \mathbb{E}^T \left(\mathbb{E}{\mathbb{M}_1}^{-1}\mathbb{E}^T \right)^{-1} \underline{f}.$$

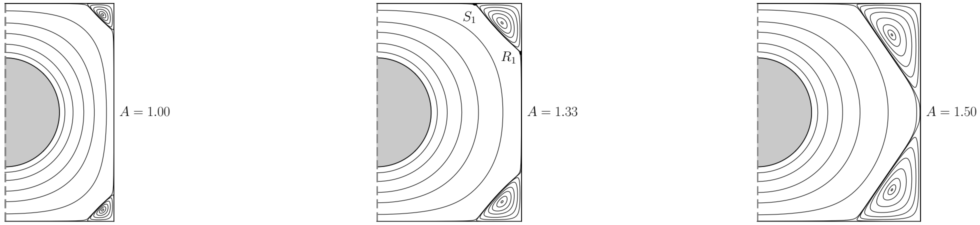
- Inverting $\mathbb{M}^{(1)}$ and $\mathbb{E}^{2,1}{\mathbb{M}^{(1)}}^{-1}{\mathbb{E}^{2,1}}^T$ is easy (in parallel) because they are element-wise-block-diagonal.
- Solving for $\underline{\check{\varphi}}$ is cheap (smaller system size and condition number). Remaining local problems for $\underline{\boldsymbol{u}}$ and $\underline{\varphi}$ are trivial.

Stokes flow induced by cylinder rotation

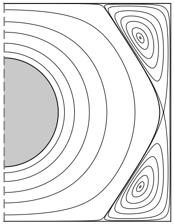


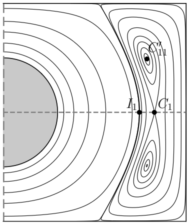
$$A = \frac{x_0}{y_0}$$

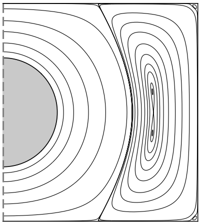
Stokes flow induced by cylinder rotation



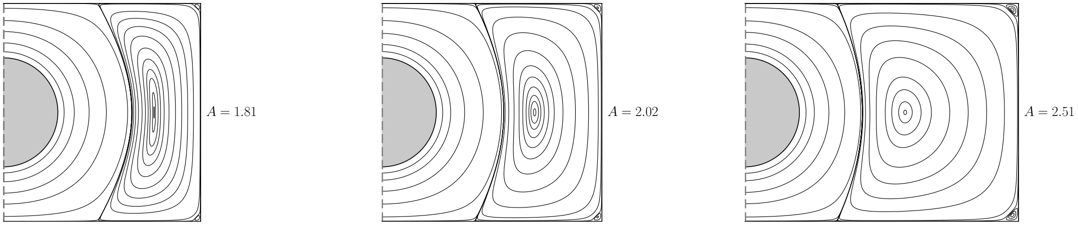
Stokes flow induced by cylinder rotation



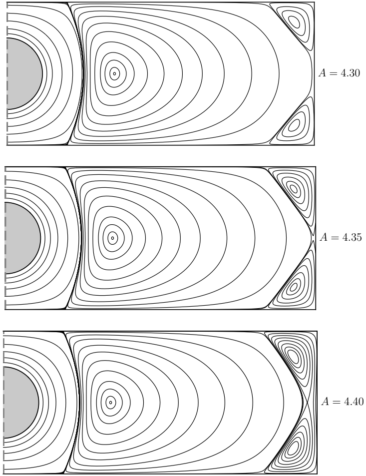




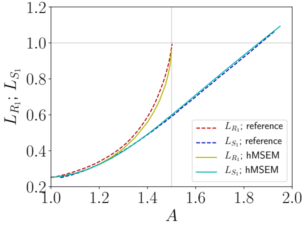
Stokes flow induced by cylinder rotation

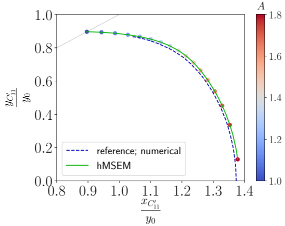


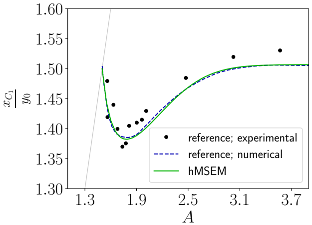
Stokes flow induced by cylinder rotation

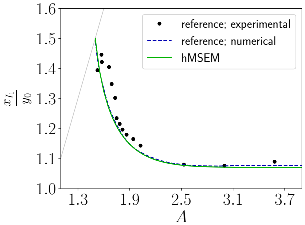


Stokes flow induced by cylinder rotation









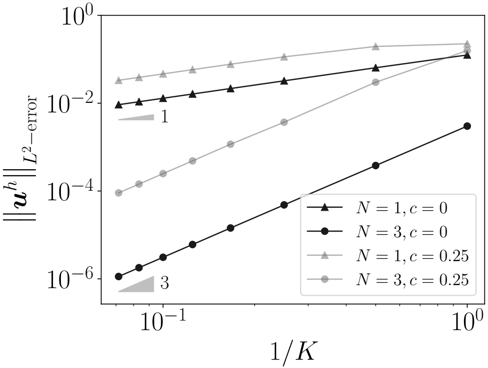
Accuracy

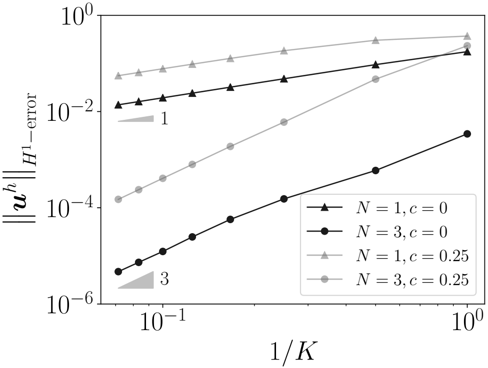
TABLE – Results of $\|x\|_{L^2\text{-error}}$ and $\|x - x'\|_{L^2\text{-norm}}$ (in brackets), where x and x' are solutions of the hMSEM and MSEM respectively, for $N \in \{1,3\}$, $K \in \{2,4,6\}$, and $c \in \{0, 0.25\}$.

x	K	$N=1$, $c=0$	$N=1$, $c=0.25$	$N=3$, $c=0$	$N=3$, $c=0.25$
u^h	2	6.4029E−2(2.05E−16)	1.9534E−1(2.79E−16)	3.8024E−4(5.57E−16)	2.9940E−2(1.70E−15)
	4	3.2265E−2(5.40E−16)	1.1353E−1(4.61E−16)	4.8312E−5(5.67E−15)	3.6850E−3(1.48E−14)
	6	2.1542E−2(7.03E−16)	7.7069E−2(8.71E−16)	1.4377E−5(4.08E−14)	1.1604E−3(1.06E−13)
ω^h	2	4.7436E−2(6.76E−16)	5.1309E−2(9.59E−16)	2.8846E−4(1.22E−14)	1.2456E−2(3.17E−14)
	4	2.3986E−2(2.98E−15)	3.8150E−2(3.31E−15)	8.2990E−5(2.63E−13)	1.0417E−3(1.00E−12)
	6	1.6008E−2(2.10E−14)	2.2952E−2(2.25E−14)	3.0156E−5(4.08E−12)	2.3494E−4(1.02E−11)
σ^h	2	9.3659E−1(1.84E−14)	2.6588(3.13E−14)	5.6919E−3(2.21E−13)	4.5920E−1(4.12E−13)
	4	4.5869E−1(8.16E−14)	1.6633(6.25E−14)	1.6879E−3(4.03E−12)	6.1945E−2(1.48E−11)
	6	3.0391E−1(2.58E−13)	1.1638(2.82E−13)	6.2626E−4(5.39E−11)	1.9624E−2(1.43E−10)

Accuracy

Super-convergence with respect to H^1-error :





Conclusions

We present the mimetic spectral element method and its hybridization with dual basis functions.

- It is mimetic/structure-preserving ; it uses nodal and integral values (with respect to nodes, edges, faces and volumes) as dof's.
- It is a hybrid/domain decomposition method ;
- It uses dual polynomials ; some discrete matrices are metric-free, extremely sparse and low order finite-difference(volume)-like (containing non-zero entries of -1 and 1 only).

Thanks a lot. Questions ?

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We present the mimetic spectral element method and its hybridization with dual basis functions.

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