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Discrete Geometries of Mathematics & Physics
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Discrete Geometries of Mathematics and Physics

Mimetic spectral element method[1]

Assignment #2

From the reference element to physical elements

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We have known how to set up mimetic spectral element spaces in the reference element $\Omega_{\mathrm{r}} = [-1,1]^2$ through Assignment #1. However, for a real problem, its computation domain most likely is different to Ω_{r}. Thus we need to know how to construct mimetic spectral element spaces in a more general element such that we can apply the mimetic spectral element method to a real problem.

1 Coordinate transformation

Consider a domain Ω_0 in the coordiante system (ξ, η) and a domain Ω in the coordinate system (x, y). A C^1 diffeomorphism (both itself and its inverse are C^1 continuous) Φ maps Ω_0 into Ω, i.e.,

$$\Phi : \Omega_0 \to \Omega,$$

by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Phi\left(\xi, \eta\right) = \begin{bmatrix} \Phi_x\left(\xi, \eta\right) \\ \Phi_y\left(\xi, \eta\right) \end{bmatrix}$$

Since the mapping Φ is C^1 continuous, we can compute its Jacobian matrix $\mathcal{J}$,

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}.$$

The Jacobian, J, of the mapping is the determinant of the Jacobian matrix, i.e.,

$$J = \det(\mathcal{J}).$$

[1] https://mathischeap.com/contents/teaching/advanced_numerical_methods/mimetic_spectral_element_method/main

The metric matrix is defined as

$$\boldsymbol{\mathcal{G}} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} := \boldsymbol{\mathcal{J}}^{\mathsf{T}}\boldsymbol{\mathcal{J}},$$

To be more exact, $g_{ij} = \sum_{m=1}^{2} \boldsymbol{\mathcal{J}}|_{mi}\, \boldsymbol{\mathcal{J}}|_{mj}$. For example, $g_{11} = \mathcal{J}_{11}\mathcal{J}_{11} + \mathcal{J}_{21}\mathcal{J}_{21}$ and $g_{12} = \mathcal{J}_{11}\mathcal{J}_{12} + \mathcal{J}_{21}\mathcal{J}_{22}$. The determinate of the metric matrix is called the metric, g, of the mapping, namely,

$$g := \det\left(\boldsymbol{\mathcal{G}}\right).$$

One can show that the metric is equal to the square of the Jacobian, i.e.,

$$g = J^2 = \left[\det\left(\boldsymbol{\mathcal{J}}\right)\right]^2.$$

As Φ is a C^1 diffeomorphism, its inverse mapping Φ^{-1},

$$\Phi^{-1} : \Omega \to \Omega_0,$$

must also be C^1 continuous. The Jacobian matrix of Φ^{-1} is called the inverse Jacobian matrix of Φ is

$$\boldsymbol{\mathcal{J}}^{-1} := \begin{bmatrix} \mathcal{J}_{11}^{-1} & \mathcal{J}_{12}^{-1} \\ \mathcal{J}_{21}^{-1} & \mathcal{J}_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}.$$

It must satisfy $\boldsymbol{\mathcal{J}}^{-1}\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{I}}$ where $\boldsymbol{\mathcal{I}}$ is the identity matrix. The inverse Jacobian is the determinant of $\boldsymbol{\mathcal{J}}^{-1}$, i.e.,

$$J^{-1} = \det\left(\boldsymbol{\mathcal{J}}^{-1}\right) = \frac{1}{J} = \frac{1}{\det\left(\boldsymbol{\mathcal{J}}\right)}.$$

The inverse metric matrix is defined as

$$\boldsymbol{\mathcal{G}}^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} := \boldsymbol{\mathcal{J}}^{-1}\left(\boldsymbol{\mathcal{J}}^{-1}\right)^{\mathsf{T}},$$

where $g^{ij} = \sum_{m=1}^{2} \boldsymbol{\mathcal{J}}^{-1}\big|_{im}\, \boldsymbol{\mathcal{J}}^{-1}\big|_{jm}$. For example, $g^{11} = \mathcal{J}_{11}^{-1}\mathcal{J}_{11}^{-1} + \mathcal{J}_{12}^{-1}\mathcal{J}_{12}^{-1}$ and $g^{12} = \mathcal{J}_{11}^{-1}\mathcal{J}_{21}^{-1} + \mathcal{J}_{12}^{-1}\mathcal{J}_{22}^{-1}$.

2 Transformation of mimetic spectral element spaces

We now consider a C^1 diffeomorphism mapping that maps the reference element into a particular element Ω_n, called the physical element, i.e.,

$$\Phi_n : \Omega_{\mathrm{r}} \to \Omega_n.$$

What we really care about is how the mimetic spectral element spaces are transformed under this mapping. Once we know it, we can set up mimetic spectral element spaces in a physical element rather than in the reference element only.

In this assignment, to limit the difficulty at a reasonable level, we will only consider the physical element to be an orthogonal (whose edges are parallel to axes) rectangle, i.e.,

$$\Omega_n := [x_0, x_1] \times [y_0, y_1],$$

where $x_0, x_1, y_0, y_1 \in \mathbb{R}$ and $x_0 < x_1$, $y_0 < y_1$. With such a physical, we know the mapping can be explicitly expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Phi_n(\xi, \eta) = \begin{bmatrix} \Phi_{n,x}(\xi) \\ \Phi_{n,y}(\eta) \end{bmatrix} = \begin{bmatrix} x_0 + \dfrac{d_x(\xi+1)}{2} \\ y_0 + \dfrac{d_y(\eta+1)}{2} \end{bmatrix},$$

where $d_x = x_1 - x_0$ and $d_y = y_1 - y_0$. And we can easily compute its metric-related values,

$$\boldsymbol{\mathcal{J}}_n = \begin{bmatrix} \frac{d_x}{2} & 0 \\ 0 & \frac{d_y}{2} \end{bmatrix},$$

$$J_n = \det(\boldsymbol{\mathcal{J}}_n) = \frac{d_x d_y}{4},$$

$$\boldsymbol{\mathcal{G}}_n = \begin{bmatrix} \frac{d_x^2}{4} & 0 \\ 0 & \frac{d_y^2}{4} \end{bmatrix},$$

$$g_n = \det(\boldsymbol{\mathcal{G}}_n) = J_n^2 = \frac{d_x^2 d_y^2}{16},$$

$$\boldsymbol{\mathcal{J}}_n^{-1} = \begin{bmatrix} \frac{2}{d_x} & 0 \\ 0 & \frac{2}{d_y} \end{bmatrix},$$

$$J_n^{-1} = \frac{4}{d_x d_y},$$

$$\boldsymbol{\mathcal{G}}_n^{-1} = \begin{bmatrix} \frac{4}{d_x^2} & 0 \\ 0 & \frac{4}{d_y^2} \end{bmatrix}.$$

Note that these metric-related values are all constants and are independenet to the origin of the rectangle, i.e. (x_0, y_0). That means if rectangles are of the same shape, they will all have same metric-related values.

To obtain transformed mimetic spectral element spaces in Ω_n, we just need to find the transformed basis functions. And if these basis functions are still linearly independent, they span the mimetic spectral element spaes in Ω_n. So, after this point, we introduce how to transform the basis functions.

2.1 Transformation: $C(\Omega_\mathrm{r}) \to C(\Omega_n)$ (or $G(\Omega_\mathrm{r}) \to G(\Omega_n)$)

We have learned from Assigment #1 that the basis functions of mimetic spectral element space $C(\Omega_\mathrm{r})$ are

$$\mathtt{ll}^{ij}(\xi,\eta) = l^i(\xi) l^j(\eta), \quad i,j \in \{0,1,\cdots,N\}.$$

We can transform them to Ω_n as

$$\mathtt{ll}_n^{ij}(x,y) = \mathtt{ll}_n^{ij}\left(\Phi\left(\xi,\eta\right)\right) = \mathtt{ll}^{ij}(\xi,\eta), \quad i,j \in \{0,1,\cdots,N\}.$$

They are a set of basis functions for the mimetic spectral element space $C(\Omega_n)$ and $G(\Omega_n)$. So, if $\omega_h \in C(\Omega_n)$ or $G(\Omega_n)$, we can express it as

$$\omega_h = \sum_{i=0}^{N} \sum_{j=0}^{N} \mathsf{w}_{ij} \mathtt{ll}_n^{ij}(x,y). \tag{1}$$

Given a function $\omega(x,y)$ in Ω_n, if $\omega_h = \pi(\omega)$, the expansion coefficients are

$$\mathsf{w}_{ij} = \omega(x_i, y_j) = \omega\left(\Phi_n\left(\xi_i, \eta_j\right)\right). \tag{2}$$

2.2 Transformation: $\boldsymbol{D}(\Omega_{\mathrm{r}}) \to \boldsymbol{D}(\Omega_n)$

We can transform basis functions of $\boldsymbol{D}(\Omega_{\mathrm{r}})$ to Ω_n by

$$\mathtt{le}_n^{ij}(x,y) = \frac{\boldsymbol{\mathcal{J}}_n|_{11}}{J_n}\mathtt{le}^{ij}\left(\Phi_n^{-1}(x,y)\right) = \frac{2}{d_y}\mathtt{le}^{ij}\left(\Phi_n^{-1}(x,y)\right) = \frac{2}{d_y}\mathtt{le}^{ij}(\xi,\eta),$$

$$\mathtt{el}_n^{ij}(x,y) = \frac{\boldsymbol{\mathcal{J}}_n|_{22}}{J_n}\mathtt{el}^{ij}\left(\Phi_n^{-1}(x,y)\right) = \frac{2}{d_x}\mathtt{el}^{ij}\left(\Phi_n^{-1}(x,y)\right) = \frac{2}{d_x}\mathtt{el}^{ij}(\xi,\eta),$$

They are basis functions that span the mimetic spectral element space $\boldsymbol{D}(\Omega_n)$. If $\boldsymbol{u}_h \in \boldsymbol{D}(\Omega_n)$, we can express it as

$$\boldsymbol{u}_h = \begin{bmatrix} \sum_{i=0}^{N}\sum_{j=1}^{N} \mathsf{u}_{ij}\mathtt{le}_n^{ij}(x,y) \\ \sum_{i=1}^{N}\sum_{j=0}^{N} \mathsf{v}_{ij}\mathtt{el}_n^{ij}(x,y) \end{bmatrix} \tag{3}$$

Given a vector $\boldsymbol{u}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ in Ω_n, if $\boldsymbol{u}_h = \pi(\boldsymbol{u})$, the expansion coefficients of $\boldsymbol{u}_h$ are

$$\begin{aligned} \mathsf{u}_{ij} &= \int_{y_{j-1}}^{y_j} u(x_i,y)\mathrm{d}y = \int_{\Phi_{n,y}(\eta_{j-1})}^{\Phi_{n,y}(\eta_j)} u\left(\Phi_{n,x}(\xi_i), y)\right)\mathrm{d}y \\ \mathsf{v}_{ij} &= \int_{x_{i-1}}^{x_i} v(x,y_j)\mathrm{d}x = \int_{\Phi_{n,x}(\xi_{i-1})}^{\Phi_{n,x}(\xi_i)} v\left(x, \Phi_{n,y}(\eta_j)\right)\mathrm{d}y \end{aligned} \tag{4}$$

2.3 Transformation: $\boldsymbol{R}(\Omega_{\mathrm{r}}) \to \boldsymbol{R}(\Omega_n)$

In orthogonal rectangle elements, the basis functions $\boldsymbol{R}(\Omega_n)$ are same to those of $\boldsymbol{D}(\Omega_n)$. But they span different components. In details, if $\boldsymbol{\sigma}_h \in \boldsymbol{R}(\Omega_n)$, we can express it as

$$\boldsymbol{\sigma}_h = \begin{bmatrix} \sum_{i=1}^{N}\sum_{j=0}^{N} \mathsf{s}_{ij}\mathtt{el}_n^{ij}(x,y) \\ \sum_{i=0}^{N}\sum_{j=1}^{N} \mathsf{t}_{ij}\mathtt{le}_n^{ij}(x,y) \end{bmatrix} \tag{5}$$

And given a vector $\boldsymbol{\sigma}(x,y) = \begin{bmatrix} s(x,y) \\ t(x,y) \end{bmatrix}$ in Ω_n, if $\boldsymbol{\sigma}_h = \pi(\boldsymbol{\sigma})$, the expansion coefficients of $\boldsymbol{\sigma}_h$ are

$$\begin{aligned} \mathsf{s}_{ij} &= \int_{x_{i-1}}^{x_i} s(x,y_j)\mathrm{d}x = \int_{\Phi_{n,x}(\xi_{i-1})}^{\Phi_{n,x}(\xi_i)} s\left(x, \Phi_{n,y}(\eta_j)\right)\mathrm{d}y \\ \mathsf{t}_{ij} &= \int_{y_{j-1}}^{y_j} t(x_i,y)\mathrm{d}y = \int_{\Phi_{n,y}(\eta_{j-1})}^{\Phi_{n,y}(\eta_j)} t\left(\Phi_{n,x}(\xi_i), y)\right)\mathrm{d}y \end{aligned} \tag{6}$$

2.4 Transformation: $S(\Omega_{\mathrm{r}}) \to S(\Omega_n)$

Fianlly, we know from Assigment #1 that the basis functions of mimetic spectral element space $S(\Omega_{\mathrm{r}})$ are

$$\mathtt{ee}^{ij}(\xi,\eta) = e^i(\xi)e^j(\eta), \quad i,j \in \{1,2,\cdots,N\}.$$

We can transform them to Ω_n as

$$\mathtt{ee}_n^{ij}(x,y) = \frac{1}{J_n}\mathtt{ee}^{ij}\left(\Phi_n^{-1}(x,y)\right) = \frac{4}{d_x d_y}\mathtt{ee}^{ij}(\xi,\eta)$$

They are a set of basis functions for the mimetic spectral element space $S(\Omega_n)$. Thus, if $\phi_h \in S(\Omega_n)$, we can express it as

$$\phi_h = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathsf{f}_{ij}\,\mathsf{ee}_n^{ij}(x,y). \tag{7}$$

Given a scalar $\phi(x,y)$ in Ω_n, if $\phi_h = \pi(\phi)$, the expansion coefficients of ϕ_h are

$$\mathsf{f}_{ij} = \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \phi(x,y)\mathrm{d}y\mathrm{d}x = \int_{\Phi_{n,x}(\xi_{i-1})}^{\Phi_{n,x}(\xi_i)}\int_{\Phi_{n,y}(\eta_{j-1})}^{\Phi_{n,y}(\eta_j)} \phi(x,y)\mathrm{d}y\mathrm{d}x. \tag{8}$$

Assignment 2.1.0: Reduction and reconstruction of $C(\Omega_n)$ and $G(\Omega_n)$

You need to program two functions to compute the reduction (2) and the reconstruction (1) for $C(\Omega_n)$ and $G(\Omega_n)$, respectively.

Assignment 2.1.1: Reduction and reconstruction of $\boldsymbol{D}(\Omega_n)$

You need to program two functions to compute the reduction (4) and the reconstruction (3) for $\boldsymbol{D}(\Omega_n)$, respectively.

Assignment 2.1.2: Reduction and reconstruction of $\boldsymbol{R}(\Omega_n)$

You need to program two functions to compute the reduction (6) and the reconstruction (5) for $\boldsymbol{R}(\Omega_n)$, respectively.

Assignment 2.1.3: Reduction and reconstruction of $S(\Omega_n)$

You need to program two functions to compute the reduction (8) and the reconstruction (7) for $S(\Omega_n)$, respectively.

Assignment 2.1.4: Incidence matrices

You now need to use the incidence matrices you have programmed in Assignmen #1 to this assignment and check out if it still works.

The answer should be "YES". This means we no need to compute new incidence matrices for the physical domain Ω_n. Think about it. Why?

3 Three dimensions

Read [1, Chapter 2] for the more details of three-dimensional cases. They can help you understanding two-dimensional cases. Here in this assignment, we only present the necessary contents for programming the functionalities. Reading more insights can help you understanding the idea and philosophy behind very much.

References

[1] Y. Zhang, Mimetic Spectral Element Method and Extensions toward Higher Computational Efficiency (2022).