Discrete Geometries of Mathematics and Physics

Mimetic spectral element method[1]

Assignment #6

Error analysis

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In previous assignments, we visually check the results. That is saying we reconstruct the solutions, plot them, and compare the plots to those of the analytical solutions. This is okay at the experimental level. But, from a certain moment, we need to quantitatively analyze the error in the results to investigate the performance of the method more accurately. This is usually called *error analysis*. In this assignment, we will together study it.

1 L^2-norm

The L^2-norm of $\alpha \in L^2(\Omega)$ and $\boldsymbol{\alpha} \in [L^2(\Omega)]^2$, is

$$\|\alpha\|_{L^2} = \sqrt{\langle \alpha, \alpha \rangle_\Omega} = \sqrt{\int_\Omega \alpha^2 \; \mathrm{d}\Omega} \; ,$$

$$\|\boldsymbol{\alpha}\|_{L^2} = \sqrt{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle_\Omega} = \sqrt{\int_\Omega \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} \; \mathrm{d}\Omega} \; .$$

As subspaces of Sobolev spaces, Hilbert spaces $H(\mathrm{curl};\Omega)$, $\boldsymbol{H}(\mathrm{div};\Omega)$, $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{rot};\Omega)$ also admit the L^2-norm. And we have following norms

$$\begin{aligned}
\|\omega\|_{H(\mathrm{curl})} &= \sqrt{\|\omega\|^2_{L^2} + \|\nabla \times \omega\|^2_{L^2}}, && \omega \in H(\mathrm{curl};\Omega), \\
\|\boldsymbol{u}\|_{H(\mathrm{div})} &= \sqrt{\|\boldsymbol{u}\|^2_{L^2} + \|\nabla \cdot \boldsymbol{u}\|^2_{L^2}}, && \boldsymbol{u} \in H(\mathrm{div};\Omega), \\
\|\phi\|_{H^1} &= \sqrt{\|\phi\|^2_{L^2} + \|\nabla \phi\|^2_{L^2}}, && b \in H^1(\Omega), \\
\|\boldsymbol{\sigma}\|_{H(\mathrm{rot})} &= \sqrt{\|\boldsymbol{\sigma}\|^2_{L^2} + \|\nabla \times \boldsymbol{\sigma}\|^2_{L^2}}, && \boldsymbol{\sigma} \in H(\mathrm{rot};\Omega).
\end{aligned}$$

[1] https://mathischeap.com/contents/teaching/advanced_numerical_methods/mimetic_spectral_element_method/main

Also, because our mimetic spectral element spaces are subspaces of them, we can use these norms to measure, for example, the outputs of our programs.

2 Error of $\boldsymbol{u}_h$ and φ_h

Let $\boldsymbol{u}_h$ and φ_h be the solutions of the Poisson problem in Assignment #4 & Assignment #5. As we have shown that we can derive the analytical solutions of the Poisson problem, i.e.,

$$\varphi = \sin(2\pi x)\sin(2\pi y),$$

$$\boldsymbol{u} = \begin{bmatrix} 2\pi\cos(2\pi x)\sin(2\pi y) \\ 2\pi\sin(2\pi x)\cos(2\pi y) \end{bmatrix}.$$

Now, suppose we have divided the domain $\Omega = [0,1]^2$ into K^2 (K is a positive integer) uniform elements. Thus, if $K = 1$, the domain is considered as a single element, which is case of Assignment #4. And if $K > 1$, it becomes the case of Assignment #5.

To measure the error in solutions $\boldsymbol{u}_h$ and φ_h, we of course can compare them to their analytical solutions $\boldsymbol{u}$ and φ. So in an element, Ω_k, we can measure the norms of $\boldsymbol{u}_h - \boldsymbol{u}$ and $\boldsymbol{\varphi}_h - \boldsymbol{\varphi}$, which indicates the error in element Ω_k. We use $H(\text{div})$-error of $\boldsymbol{u}_h$ and the L^2-error of φ_h. In the element Ω_k, they are

(1) $$\|\boldsymbol{u}_k\|_{H(\text{div})\text{-error}} = \|\boldsymbol{u}_k - \boldsymbol{u}\|_{H\text{-div}},$$

(2) $$\|\varphi_k\|_{L^2\text{-error}} = \|\varphi_k - \varphi\|_{L^2}.$$

And in the whole domain Ω, the $H(\text{div})$-error of $\boldsymbol{u}_h$ and the L^2-error of φ_h are

(3a) $$\|\boldsymbol{u}_h\|_{H(\text{div})\text{-error}} = \sqrt{\sum_{k=1}^{K^2} \|\boldsymbol{u}_k\|^2_{H(\text{div})\text{-error}}},$$

(3b) $$\|\varphi_h\|_{L^2\text{-error}} = \sqrt{\sum_{k=1}^{K^2} \|\varphi_k\|^2_{L^2\text{-error}}}.$$

3 Computation of error in one element

Now, we can see that the key is how to compute the $H(\text{div})$-error of $\boldsymbol{u}_h$ and the L^2-error of φ_h in an element Ω_k, i.e., (1) and (2). We take (1) as an example and have a look at its details;

$$\begin{aligned}
\|\boldsymbol{u}_k\|_{H(\text{div})\text{-error}} &= \|\boldsymbol{u}_k - \boldsymbol{u}\|_{H\text{-div}} \\
&= \sqrt{\|\boldsymbol{u}_k - \boldsymbol{u}\|^2_{L^2} + \|\nabla\cdot\boldsymbol{u}_k - \nabla\cdot\boldsymbol{u}\|^2_{L^2}} \\
&= \sqrt{\|\boldsymbol{u}_k - \boldsymbol{u}\|^2_{L^2} + \|-f_k - \nabla\cdot\boldsymbol{u}\|^2_{L^2}} \\
&= \sqrt{\|\boldsymbol{u}_k - \boldsymbol{u}\|^2_{L^2} + \|f_k + \nabla\cdot\boldsymbol{u}\|^2_{L^2}},
\end{aligned}$$

where f_k is the project of given function f to Ω_k, i.e., $f_k = \pi(f)$. In other words, the mimetic spectral element method can make sure that the solution satisfies $-\nabla\cdot\boldsymbol{u}_k = f_k$. From this equation and (2), we can see that we just need to compute the L^2-norm of something in an element. Then we can quickly compute the errors. We now use $\|\boldsymbol{u}_k - \boldsymbol{u}\|^2_{L^2}$ as a demonstration,

$$\|\boldsymbol{u}_k - \boldsymbol{u}\|^2_{L^2} = \langle \boldsymbol{u}_k - \boldsymbol{u}, \boldsymbol{u}_k - \boldsymbol{u}\rangle_{\Omega_k}.$$

Recall the numerical integration with Gauss quadrature as we have learned in Assignment #3. Given a quadrature degree N_q, let

$$\left\{x_1, x_2, \cdots, x_{N_q}\right\}, \quad \left\{y_1, y_2, \cdots, y_{N_q}\right\}, \quad \left\{w_1, w_2, \cdots, w_{N_q}\right\},$$

be the Gauss sample nodes along two axes and Gauss weights. Then we have

$$\langle \boldsymbol{u}_k - \boldsymbol{u}, \boldsymbol{u}_k - \boldsymbol{u} \rangle_{\Omega_k} \approx \sum_{i=1}^{N_q} \sum_{j=1}^{N_q} \frac{h^2}{4} w_i w_j \left(\boldsymbol{u}_k(x_i, y_j) - \boldsymbol{u}(x_i, y_j)\right)^2. \tag{4}$$

In the right hand side of this equation, everything is very explicit except $\boldsymbol{u}_k(x_i, y_j)$ which is in fact the reconstruction of $\boldsymbol{u}_k$ at the Gauss sample nodes. If you re-visit the reconstruction function you have programmed, you will be able to program the function of (4).

Then, you should be able compute the error in one element Ω_k. And then, using (3), we can compute the error in the whole domain Ω.

Assignment 6.1.0: Program it!

Program to compute the error of your solutions in Assignment #5. Compute the error for different K and N. You can try to find out how the error decreases when we increase K or N.