Research group
Discrete Geometries of Mathematics & Physics
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Discrete Geometries of Mathematics and Physics

Mimetic spectral element method[1]

Assignment #4

Weak formulation and discretization: Single element

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So far, we have built all ingredients necessary for solving a problem. In this assignment, we will use them to solve a practical problem.

1 Poisson equation

The Poisson equation is a classic model of a large range of physical problems. It is one of the most important PDE's in computational physics.

In a unit square domain $\Omega = [0,1]^2$, we consider a most simple form of the Poisson equation, an elliptical equation written as

$$-\Delta\varphi = f, \tag{1}$$

where φ and f are two scalar fields, and Δ is the scalar Laplacian operator, i.e., $\Delta = \nabla\cdot\nabla$. If we introduce an intermediate variable $\boldsymbol{u} := \nabla\varphi$, the Poisson equation (1) can be re-written into a mixed form,

$$\boldsymbol{u} = \nabla\varphi, \tag{2a}$$

$$-\nabla\cdot\boldsymbol{u} = f. \tag{2b}$$

Given f and a proper boundary condition, the problem is closed. In this assignment, we only consider the homogeneous boundary condition $\varphi = 0$ over the whole boundary $\partial\Omega$.

[1] https://mathischeap.com/contents/teaching/advanced_numerical_methods/mimetic_spectral_element_method/main

2 Weak formulation

To explore the weak formulation of (2), we firstly restrict the variables to Sobolev spaces, $(\varphi, \boldsymbol{u}, f) \in L^2(\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$. Note that this choice is according to Hilbert complexes

$$0 \longrightarrow H(\mathrm{curl};\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \longrightarrow 0,$$

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{rot};\Omega) \xrightarrow{\nabla\times} L^2(\Omega) \longrightarrow 0.$$

In (2b), the divergence operator is applied to $\boldsymbol{u}$. Thus, a natural choice is $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div};\Omega)$. However, (2a) says $\boldsymbol{u} = \nabla\varphi$. If we choose $\varphi \in H^1(\Omega)$, $\boldsymbol{u} = \nabla\varphi$ will be in $\boldsymbol{H}(\mathrm{rot};\Omega)$ which is different from $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div};\Omega)$. To overcome this contradictory, we will need to use the integration by parts to convert the gradient operator to a divergence operator. As a result, put φ in $L^2(\Omega)$ becomes fine. To see this more clearly, we test (2a) with $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and test (2b) with $\psi \in L^2(\Omega)$. We obtain

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\Omega - \langle \nabla\varphi, \boldsymbol{v}\rangle_\Omega = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega), \tag{3a}$$

$$-\langle \nabla\cdot\boldsymbol{u}, \psi\rangle_\Omega = \langle f, \psi\rangle_\Omega, \qquad \forall \psi \in L^2(\Omega). \tag{3b}$$

Do not forget the sign $\forall$ means *FOR ALL*. Since $\varphi \in L^2(\Omega)$, it does not admit a gradient operator (or the gradient of φ is not guaranteed L^2-integrable). Thus, we should use the integration by parts to it,

$$\langle \nabla\varphi, \boldsymbol{v}\rangle_\Omega = -\langle \varphi, \nabla\cdot\boldsymbol{v}\rangle_\Omega + \int_{\partial\Omega} \varphi\,(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}\Gamma,$$

where $\boldsymbol{n}$ is the unit outward norm vector. And, because in this assignment, we use homogeneous boundary condition $\varphi = 0$ all over the boundary, we must have

$$\int_{\partial\Omega} \varphi\,(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}\Gamma = 0.$$

Thus, we finally get

$$\langle \nabla\varphi, \boldsymbol{v}\rangle_\Omega = -\langle \varphi, \nabla\cdot\boldsymbol{v}\rangle_\Omega\,.$$

Use this relation to the second term of (3a), we finally get the weak formulation for this problem: Given $f \in L^2(\Omega)$, seek $(\boldsymbol{u}, \varphi) \in \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega)$, such that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\Omega + \langle \varphi, \nabla\cdot\boldsymbol{v}\rangle_\Omega = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega), \tag{4a}$$

$$-\langle \nabla\cdot\boldsymbol{u}, \psi\rangle_\Omega = \langle f, \psi\rangle_\Omega, \qquad \forall \psi \in L^2(\Omega). \tag{4b}$$

This way of deriving the weak formulation (4) in fact is not strictly mathematically sound, but it is a good approach to help the ones new to this method understanding it better.

3 Discretization

We see that the final weak formulation (4) is still at the continuous level. We now need to use discrete spaces to approximate the Sobolev spaces there. For now, we consider the whole domain, $\Omega = [0,1]^2$, to one element. Therefore, we can quickly construct mimetic spectral element spaces in this physical element Ω,

$$0 \longrightarrow C(\Omega) \xrightarrow{\nabla\times} \boldsymbol{D}(\Omega) \xrightarrow{\nabla\cdot} S(\Omega) \longrightarrow 0, \tag{5a}$$

$$0 \longrightarrow G(\Omega) \xrightarrow{\nabla} \boldsymbol{R}(\Omega) \xrightarrow{\nabla\times} S(\Omega) \longrightarrow 0. \tag{5b}$$

Using these mimetic spectral element spaces to approximate these Sobolev spaces in (4), we obtain a discrete version of (3): Given $f_h \in S(\Omega)$, seek $(\boldsymbol{u}_h, \varphi_h) \in \boldsymbol{D}(\Omega) \times S(\Omega)$, such that

(6a) $$\langle \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \varphi_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{D}(\Omega),$$

(6b) $$-\langle \nabla \cdot \boldsymbol{u}_h, \psi_h \rangle_\Omega = \langle f_h, \psi_h \rangle_\Omega , \qquad \forall \psi_h \in S(\Omega).$$

Now, we shall recall the incidence and mass matrices we have learned and programmed in previous assignments. We can re-write (6) into a matrix format as follows. Given $f_h \in S(\Omega)$, seek $(\boldsymbol{u}_h, \varphi_h) \in \boldsymbol{D}(\Omega) \times S(\Omega)$, such that

(7a) $$\vec{\boldsymbol{v}}^\mathsf{T} \mathbb{M}_D \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}^\mathsf{T} \mathbb{E}_D^\mathsf{T} \mathbb{M}_S \vec{\varphi} = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{D}(\Omega),$$

(7b) $$-\vec{\psi}^\mathsf{T} \mathbb{M}_S \mathbb{E}_D \vec{\boldsymbol{u}} = \vec{\psi}^\mathsf{T} \mathbb{M}_S \vec{f}, \qquad \forall \psi_h \in S(\Omega),$$

Since it holds for all $\boldsymbol{v}_h \in \boldsymbol{D}(\Omega)$ and $\psi_h \in S(\Omega)$, we can, for example, select $\boldsymbol{v}_h$ such that $\vec{\boldsymbol{v}}$ to be

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

consecutively. As a result, (7a) becomes

$$\mathbb{M}_D \vec{\boldsymbol{u}} + \mathbb{E}_D^\mathsf{T} \mathbb{M}_S \vec{\varphi} = \mathbf{0}.$$

Similarly, (7b) can become

(8) $$-\mathbb{M}_S \mathbb{E}_D \vec{\boldsymbol{u}} = \mathbb{M}_S \vec{f}.$$

Then, we eventually get a linear system to be solved,

$$\begin{aligned} \mathbb{M}_D \vec{\boldsymbol{u}} + \mathbb{E}_D^\mathsf{T} \mathbb{M}_S \vec{\varphi} &= \mathbf{0}, \\ -\mathbb{M}_S \mathbb{E}_D \vec{\boldsymbol{u}} &= \mathbb{M}_S \vec{f}, \end{aligned}$$

which can be expressed in a matrix equation,

(9) $$\begin{bmatrix} \mathbb{M}_D & \mathbb{E}_D^\mathsf{T} \mathbb{M}_S \\ -\mathbb{M}_S \mathbb{E}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbb{M}_S \vec{f} \end{bmatrix},$$

which is a typical $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ matrix equation. Recall that the mass matrix $\mathbb{M}_S$ is symmetric (in fact, all mass matrices are symmetric), i.e. $\mathbb{M}_S^\mathsf{T} = \mathbb{M}_S$. Thus, we know $\left(\mathbb{E}_D^\mathsf{T} \mathbb{M}_S\right)^\mathsf{T} = \mathbb{M}_S^\mathsf{T} \mathbb{E}_D = \mathbb{M}_S \mathbb{E}_D$. This shows the total left matrix of (9), i.e. the $\boldsymbol{A}$ matrix, is anti-symmetric, $\boldsymbol{A}^\mathsf{T} = -\boldsymbol{A}$. Or if you apply a minus to (8), the final linear system becomes

(10) $$\begin{bmatrix} \mathbb{M}_D & \mathbb{E}_D^\mathsf{T} \mathbb{M}_S \\ \mathbb{M}_S \mathbb{E}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathbb{M}_S \vec{f} \end{bmatrix},$$

where the $\boldsymbol{A}$ matrix (of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ form) now becomes symmetric. Furthermore, (8) can be simplified as

$$\mathbb{E}_D \vec{\boldsymbol{u}} = -\vec{f}$$

if we left-multiply both sides of (8) by the inverse of the mass matrix (a mass matrix must be invertible). Therefore, we can get a simpler linear system

(11) $$\begin{bmatrix} \mathbb{M}_D & \mathbb{E}_D^\mathsf{T} \mathbb{M}_S \\ \mathbb{E}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\vec{f} \end{bmatrix}.$$

Now, if we send (9), (10), or (11) to a linear system solver, we can obtain $\vec{\boldsymbol{u}}$ and $\vec{\varphi}$. And, using them we can reconstruct $\boldsymbol{u}_h$ and φ_h which are the solutions approximating $\boldsymbol{u}$ and φ.

4 A real problem

Now, we consider a real problem. Still, the domain is $\Omega = [-1, 1]^2$. And f is given as

$$f = 8\pi^2 \sin(2\pi x) \sin(2\pi y). \tag{12}$$

For a mimetic spectral element degree N, we can obtain

- incidence and mass matrices by calling the functions you have programmed in previous assignment;
- the vector $\vec{f}$ by calling the reduction function for $S(\Omega)$ you have programmed;

Do not forget that, in this assignment, we consider the whole domain $\Omega = [-1, 1]^2$ as a single physical element.

Using the matrices and $\vec{f}$, we can build $\boldsymbol{A}$ and $\boldsymbol{b}$ of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ according to (9), (10), or (11). Then, we can send $\boldsymbol{A}$ and $\boldsymbol{b}$ to a linear system solver, for example, see `numpy.linalg.solve` (you do not want to program a linear system solver by yourself, right?), and solve for $\boldsymbol{x} = \begin{bmatrix} \vec{\boldsymbol{u}} & \vec{\varphi} \end{bmatrix}^\mathsf{T}$.

Assignment 4.1.0: Program it!

Program it to solve the problem above. And reconstruct the solutions $\boldsymbol{u}_h$ and φ_h using the output of the linear solver $\vec{\boldsymbol{u}}$ and $\vec{\varphi}$.

For the given f in (12) and the boundary condition $\varphi = 0$ on $\partial\Omega$, we actually can find the analytical solutions of φ and $\boldsymbol{u}$,

$$\varphi = \sin(2\pi x) \sin(2\pi y),$$

$$\boldsymbol{u} = \begin{bmatrix} 2\pi \cos(2\pi x) \sin(2\pi y) \\ 2\pi \sin(2\pi x) \cos(2\pi y) \end{bmatrix}.$$

By comparing your reconstructed solutions $\boldsymbol{u}_h$ and φ_h to them, you can check whether your program is correct. Try it using, for example, $N \in \{3, 4, 5, \cdots, 12\}$.