A new MEEVC discretization for two-dimensional incompressible Navier-Stokes equations

Yi Zhang[√], Artur Palha[♭], Marc Gerritsma[♭], Qinghe Yao[♯]

[√]zhangyi_aero@hotmail.com

[√]University of Twente
[♭]Delft University of Technology
[♯]Sun Yat-sen University

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ROBOTICS AND MECHATRONICS

The rotational form

In a space-time domain, the dimensionless rotational or Lamb form of 2d incompressible Navier-Stokes equations is

$$\begin{aligned} \partial_t \boldsymbol{u} + \omega \times \boldsymbol{u} + \mathrm{Re}^{-1} \nabla \times \omega + \nabla P = \boldsymbol{f} \quad & \text{in } \Omega \times (0, T], \\ \omega - \nabla \times \boldsymbol{u} = \boldsymbol{0} \quad & \text{in } \Omega \times (0, T], \\ \nabla \cdot \boldsymbol{u} = 0 \quad & \text{in } \Omega \times (0, T], \end{aligned}$$

which is supplemented with an initial condition,

$$\boldsymbol{u}^0 = \boldsymbol{u}(x, t_0),$$

and two pairs of boundary conditions,

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \widehat{u}_\perp & \text{on } \Gamma_\perp \times (0, T] \\ P = \widehat{P} & \text{on } \Gamma_{\widehat{P}} \times (0, T] \end{cases}, \qquad \begin{cases} \omega = \widehat{\omega} & \text{on } \Gamma_{\widehat{\omega}} \times (0, T] \\ \boldsymbol{u} \times \boldsymbol{n} = \widehat{u}_\parallel & \text{on } \Gamma_\parallel \times (0, T] \end{cases},$$

where $\Gamma_\perp \cap \Gamma_{\widehat{P}} = \Gamma_{\widehat{\omega}} \cap \Gamma_\parallel = \emptyset$, and $\Gamma_\perp \cup \Gamma_{\widehat{P}} = \Gamma_{\widehat{\omega}} \cup \Gamma_\parallel = \partial\Omega$.

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MEEVC

"**mass, energy ($\mathcal{K}$), enstrophy ($\mathcal{E}$) and vorticity ($\mathcal{W}$) conserving (MEEVC)**"

$$\mathcal{K} := \frac{1}{2}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}\Omega, \quad \mathcal{E} := \frac{1}{2}\int_\Omega \omega\cdot\omega\,\mathrm{d}\Omega$$

The divergence free condition of velocity,

$$\nabla\cdot\boldsymbol{u} = 0,$$

implies mass conservation. In the absence of external forces, i.e., $\boldsymbol{f} = \boldsymbol{0}$, and no net flux,

$$\partial_t\mathcal{K} = -2\mathrm{Re}^{-1}\mathcal{E},$$

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and, total vorticity,

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The original MEEVC scheme [1]

Find $\boldsymbol{u}_h \in \mathrm{RT}_N$, $\omega_h \in \mathrm{CG}_N$, and $P_h \in \mathrm{DG}_{N-1}$, such that

$$\begin{cases} \langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = -\nu \langle \nabla \times \omega_h, \boldsymbol{v}_h \rangle_\Omega , \\ \langle \partial_t \omega_h, \xi_h \rangle_\Omega - \frac{1}{2} \langle \omega_h, \nabla \cdot (\boldsymbol{u}_h \xi_h) \rangle_\Omega + \frac{1}{2} \langle \nabla \cdot (\boldsymbol{u}_h \omega_h), \xi_h \rangle_h = -\nu \langle \nabla \times \omega_h, \nabla \times \xi_h \rangle_\Omega , \\ \langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0, \end{cases}$$

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Time staggering of the two evolution equations.

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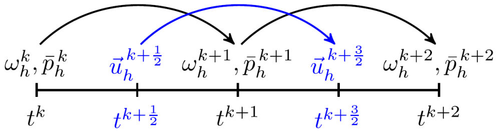

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A single evolution equation formulation

Consider finite dimensional Sobolev spaces,

$$C(\Omega) \subset H(\mathrm{curl};\Omega), \quad D(\Omega) \subset H(\mathrm{div};\Omega), \quad S(\Omega) \subset L^2(\Omega),$$

that form a discrete de Rham complex,

$$\mathbb{R} \hookrightarrow C(\Omega) \xrightarrow{\nabla\times} D(\Omega) \xrightarrow{\nabla\cdot} S(\Omega) \to 0.$$

Given $f \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega,\Gamma_\perp) \times C_0(\Omega,\Gamma_{\widehat{\omega}}) \times S(\Omega)$,

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subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_\perp \in \mathcal{T}D(\Omega,\Gamma_\perp)$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega,\Gamma_{\widehat{\omega}})$, and initial conditions $(\boldsymbol{u}_h^0, \omega_h^0) \in D(\Omega) \times C(\Omega)$.

NTE.

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NTE.

Dissipation and conservation properties

- the domain is periodic ($\partial\Omega = \emptyset$)
- no external force.

Mass conservation : $\boldsymbol{u}_h$ is selected to be in $D(\Omega) \subset H(\mathrm{div};\Omega)$,

$$\nabla \cdot \boldsymbol{u}_h = 0.$$

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega,\Gamma_\perp) \times C_0(\Omega,\Gamma_{\widehat{\omega}}) \times S(\Omega)$,

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Energy dissipation and conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega,\Gamma_{\perp}) \times C_0(\Omega,\Gamma_{\widehat{\omega}}) \times S(\Omega)$,

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If we replace $\boldsymbol{v}_h$ by $\boldsymbol{u}_h \in D(\Omega)$, we obtain

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which leads to (semi-)discrete energy balance :

$$\partial_t \mathcal{K}_h = \langle \partial_t \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_\Omega = -\mathrm{Re}^{-1} \langle \omega_h, \omega_h \rangle_\Omega = -2\mathrm{Re}^{-1} \mathcal{E}_h.$$

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$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \boldsymbol{u}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{u}_h \rangle_\Omega = 0,$$

which leads to (semi-)discrete energy balance :

$$\partial_t \mathcal{K}_h = \langle \partial_t \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_\Omega = -\mathrm{Re}^{-1} \langle \omega_h, \omega_h \rangle_\Omega = -2\mathrm{Re}^{-1} \mathcal{E}_h.$$

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Enstrophy dissipation and conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot}; \Omega, \Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega, \Gamma_{\perp}) \times C_0(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \boldsymbol{v}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T} \boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$\langle \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega - \langle \omega_h, \xi_h \rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T} \xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega, \Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $(\boldsymbol{u}_h^0, \omega_h^0) \in D(\Omega) \times C(\Omega)$.

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$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \boldsymbol{v}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T} \boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$\langle \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega - \langle \omega_h, \xi_h \rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T} \xi_h \right\rangle_{\Gamma_{\parallel}},$$

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subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega, \Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega) \times C(\Omega)$.

We know that, $\forall\ \xi_h \in C(\Omega)$, the velocity evolution equation must hold for $\nabla \times \xi_h \in D(\Omega)$, namely,

$$\langle \partial_t \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \nabla \times \xi_h \rangle_\Omega - \langle P_h, \nabla \cdot \nabla \times \xi_h \rangle_\Omega = 0,$$

If we take the time derivative of vorticity equation, we obtain

$$\langle \partial_t \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega = \langle \partial_t \omega_h, \xi_h \rangle_\Omega, \quad \forall \xi_h \in C(\Omega).$$

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Enstrophy dissipation and conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\text{rot}; \Omega, \Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega, \Gamma_{\perp}) \times C_0(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \text{Re}^{-1} \langle \nabla \times \omega_h, \boldsymbol{v}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T} \boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$\langle \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega - \langle \omega_h, \xi_h \rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T} \xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T} \boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega, \Gamma_{\perp})$ and $\mathcal{T} \omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $(\boldsymbol{u}_h^0, \omega_h^0) \in D(\Omega) \times C(\Omega)$.

We know that, $\forall\, \xi_h \in C(\Omega)$, the velocity evolution equation must hold for $\nabla \times \xi_h \in D(\Omega)$, namely,

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If we take the time derivative of vorticity equation, we obtain

$$\langle \partial_t \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega = \langle \partial_t \omega_h, \xi_h \rangle_\Omega, \quad \forall \xi_h \in C(\Omega).$$

Enstrophy dissipation and conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega, \Gamma_\perp) \times C_0(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h\rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h\rangle_\Omega + \mathrm{Re}^{-1}\langle \nabla\times\omega_h, \boldsymbol{v}_h\rangle_\Omega - \langle P_h, \nabla\cdot\boldsymbol{v}_h\rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T}\boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$\langle \boldsymbol{u}_h, \nabla\times\xi_h\rangle_\Omega - \langle \omega_h, \xi_h\rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T}\xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla\cdot\boldsymbol{u}_h, q_h\rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_\perp \in \mathcal{T}D(\Omega,\Gamma_\perp)$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega)\times C(\Omega)$.

$$\langle \partial_t \omega_h, \xi_h\rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla\times\xi_h\rangle_\Omega + \mathrm{Re}^{-1}\langle \nabla\times\omega_h, \nabla\times\xi_h\rangle_\Omega = 0, \quad \forall \xi_h \in C(\Omega)\,.$$

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Enstrophy dissipation and conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot}; \Omega, \Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega, \Gamma_{\perp}) \times C_0(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \boldsymbol{v}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T} \boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$\langle \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega - \langle \omega_h, \xi_h \rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T} \xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega, \Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega) \times C(\Omega)$.

$$\langle \partial_t \omega_h, \xi_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \nabla \times \xi_h \rangle_\Omega = 0, \quad \forall \xi_h \in C(\Omega).$$

We can replace ξ_h by $\omega_h \in C(\Omega)$ and get

$$\langle \partial_t \omega_h, \omega_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla \times \omega_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \nabla \times \omega_h \rangle_\Omega = 0.$$

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Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega,\Gamma_{\perp}) \times C_0(\Omega,\Gamma_{\widehat{\omega}}) \times S(\Omega)$,

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$$\langle \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega - \langle \omega_h, \xi_h \rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T}\xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega,\Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega,\Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega) \times C(\Omega)$.

$$\langle \partial_t \omega_h, \xi_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \nabla \times \xi_h \rangle_\Omega = 0, \quad \forall \xi_h \in C(\Omega)\,.$$

We can replace ξ_h by $\omega_h \in C(\Omega)$ and get

$$\langle \partial_t \omega_h, \omega_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla \times \omega_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \nabla \times \omega_h \rangle_\Omega = 0\,.$$

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Enstrophy dissipation and conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega, \Gamma_{\perp}) \times C_0(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\left\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \right\rangle_\Omega + \left\langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \right\rangle_\Omega + \mathrm{Re}^{-1} \left\langle \nabla \times \omega_h, \boldsymbol{v}_h \right\rangle_\Omega - \left\langle P_h, \nabla \cdot \boldsymbol{v}_h \right\rangle_\Omega = \left\langle \boldsymbol{f}, \boldsymbol{v}_h \right\rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T}\boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

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$$\left\langle \nabla \cdot \boldsymbol{u}_h, q_h \right\rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega, \Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega) \times C(\Omega)$.

As $\boldsymbol{u}_h \in D(\Omega)$ and $\nabla \cdot \boldsymbol{u}_h = 0$ is satisfied pointwise, we can find a stream function $\psi_h \in C(\Omega)$ such that $\boldsymbol{u}_h = \nabla \times \psi_h$, thus

$$\omega_h \times \boldsymbol{u}_h = \omega_h \times \nabla \times \psi_h = \nabla\left(\omega_h \psi_h\right) - \psi_h \times \nabla \times \omega_h.$$

Therefore,

$$\begin{aligned}\left\langle \omega_h \times \boldsymbol{u}_h, \nabla \times \omega_h \right\rangle_\Omega &= \left\langle \nabla\left(\omega_h \psi_h\right), \nabla \times \omega_h \right\rangle_\Omega - \left\langle \psi_h \times \nabla \times \omega_h, \nabla \times \omega_h \right\rangle_\Omega \\ &= \left\langle \omega_h \psi_h, \nabla \cdot \nabla \times \omega_h \right\rangle_\Omega - \left\langle \psi_h \times \nabla \times \omega_h, \nabla \times \omega_h \right\rangle_\Omega\end{aligned}$$

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$$\langle \boldsymbol{u}_h, \nabla \times \xi_h \rangle_\Omega - \langle \omega_h, \xi_h \rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T} \xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega, \Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega, \Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega) \times C(\Omega)$.

As $\boldsymbol{u}_h \in D(\Omega)$ and $\nabla \cdot \boldsymbol{u}_h = 0$ is satisfied pointwise, we can find a stream function $\psi_h \in C(\Omega)$ such that $\boldsymbol{u}_h = \nabla \times \psi_h$, thus

$$\omega_h \times \boldsymbol{u}_h = \omega_h \times \nabla \times \psi_h = \nabla \left(\omega_h \psi_h\right) - \psi_h \times \nabla \times \omega_h.$$

Therefore,

$$\begin{aligned} \langle \omega_h \times \boldsymbol{u}_h, \nabla \times \omega_h \rangle_\Omega &= \langle \nabla \left(\omega_h \psi_h\right), \nabla \times \omega_h \rangle_\Omega - \langle \psi_h \times \nabla \times \omega_h, \nabla \times \omega_h \rangle_\Omega \\ &= \langle \omega_h \psi_h, \nabla \cdot \nabla \times \omega_h \rangle_\Omega - \langle \psi_h \times \nabla \times \omega_h, \nabla \times \omega_h \rangle_\Omega \end{aligned}$$

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Given $\boldsymbol{f} \in [L^2(\Omega)]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega,\Gamma_{\perp}) \times C_0(\Omega,\Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_\Omega + \mathrm{Re}^{-1} \langle \nabla \times \omega_h, \boldsymbol{v}_h \rangle_\Omega - \langle P_h, \nabla \cdot \boldsymbol{v}_h \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T} \boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

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$$\langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega,\Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega,\Gamma_{\widehat{\omega}})$, and initial conditions $(\boldsymbol{u}_h^0, \omega_h^0) \in D(\Omega) \times C(\Omega)$.

We get a (semi-)discrete enstrophy balance :

$$\langle \partial_t \omega_h, \omega_h \rangle_\Omega = -\mathrm{Re}^{-1} \langle \nabla \times \omega_h, \nabla \times \omega_h \rangle_\Omega = -2\mathrm{Re}^{-1} \mathcal{P}_h \,.$$

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Vorticity conservation

Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^2$ and natural boundary conditions, $\widehat{P} \in H^{1/2}(\Omega, \Gamma_{\widehat{P}})$ and $\widehat{u}_{\parallel} \in \mathcal{T}H(\mathrm{rot};\Omega,\Gamma_{\parallel})$, seek $(\boldsymbol{u}_h, \omega_h, P_h) \in D(\Omega) \times C(\Omega) \times S(\Omega)$, such that, $\forall\, (\boldsymbol{v}_h,\ \xi_h,\ q_h) \in D_0(\Omega,\Gamma_{\perp}) \times C_0(\Omega,\Gamma_{\widehat{\omega}}) \times S(\Omega)$,

$$\langle \partial_t \boldsymbol{u}_h, \boldsymbol{v}_h\rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \boldsymbol{v}_h\rangle_\Omega + \mathrm{Re}^{-1}\langle \nabla\times\omega_h, \boldsymbol{v}_h\rangle_\Omega - \langle P_h, \nabla\cdot\boldsymbol{v}_h\rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle_\Omega - \left\langle \widehat{P} \middle| \mathcal{T}\boldsymbol{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$\langle \boldsymbol{u}_h, \nabla\times\xi_h\rangle_\Omega - \langle \omega_h, \xi_h\rangle_\Omega = \left\langle \widehat{u}_{\parallel} \middle| \mathcal{T}\xi_h \right\rangle_{\Gamma_{\parallel}},$$

$$\langle \nabla\cdot\boldsymbol{u}_h, q_h\rangle_\Omega = 0,$$

subject to essential boundary conditions, $\mathcal{T}\boldsymbol{u}_h = \widehat{u}_{\perp} \in \mathcal{T}D(\Omega,\Gamma_{\perp})$ and $\mathcal{T}\omega_h = \widehat{\omega} \in \mathcal{T}C(\Omega,\Gamma_{\widehat{\omega}})$, and initial conditions $\left(\boldsymbol{u}_h^0, \omega_h^0\right) \in D(\Omega)\times C(\Omega)$.

From

$$\langle \partial_t \omega_h, \xi_h\rangle_\Omega + \langle \omega_h \times \boldsymbol{u}_h, \nabla\times\xi_h\rangle_\Omega + \mathrm{Re}^{-1}\langle \nabla\times\omega_h, \nabla\times\xi_h\rangle_\Omega = 0, \quad \forall \xi_h \in C(\Omega),$$

we obtain, by taking $\xi_h = 1$,

$$\partial_t \mathcal{W}_h = \langle \partial_t\omega_h, 1\rangle_\Omega = 0,$$

$$\mathcal{W}_h \equiv 0.$$

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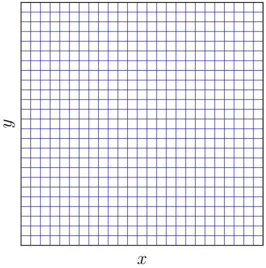
Discretization

Implicit midpoint temporal discretization ; mimetic spectral elements ; **orthogonal and curvilinear meshes**

$$\begin{cases} x = \alpha\left(r + \frac{1}{2}c\sin(2\pi r)\sin(2\pi s)\right) \\ y = \alpha\left(s + \frac{1}{2}c\sin(2\pi r)\sin(2\pi s)\right) \end{cases},$$



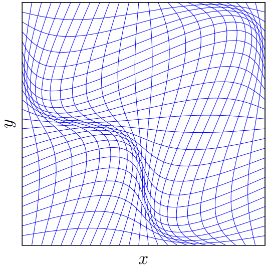


Figure – Meshes for $K = 25$, $c = 0$ (left) and $c = 0.25$ (right).

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Accuracy test : Taylor–Green vortex

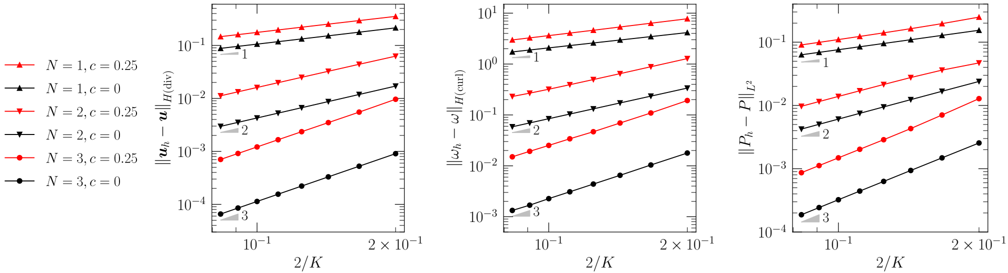

Figure – $H(\mathrm{div})$-error of $\boldsymbol{u}_h$, $H(\mathrm{curl})$-error of ω_h and L^2-error of P_h at $t = 1$ of the Taylor–Green vortex test under *ph*-refinements for $N \in \{1, 2, 3\}$, $c \in \{0, 0.25\}$, $K \in \{10, 12, 14 \cdots, 24\}$, $\Delta t = \frac{1}{25}$ and $\mathrm{Re} = 100$.

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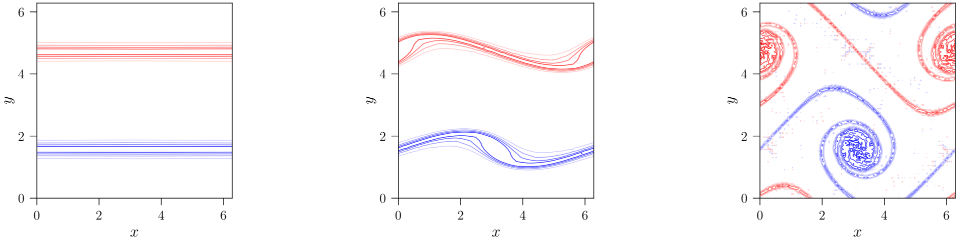
Conservation and dissipation tests : Shear layer roll-up

The shear layer roll-up is a two-dimensional ideal flow whose initial condition is given by

$$u^0 = \begin{cases} \tanh\left(\dfrac{y-\pi/2}{\delta}\right), & y \leq \pi \\ \tanh\left(\dfrac{3\pi/2-y}{\delta}\right), & y > \pi \end{cases}, \qquad v^0 = \epsilon \sin(x),$$

where $\delta = \frac{\pi}{15}$ and $\epsilon = 0.05$



(a) $t = 0$ (b) $t = 4$ (c) $t = 8$

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Conservation and dissipation tests : Shear layer roll-up

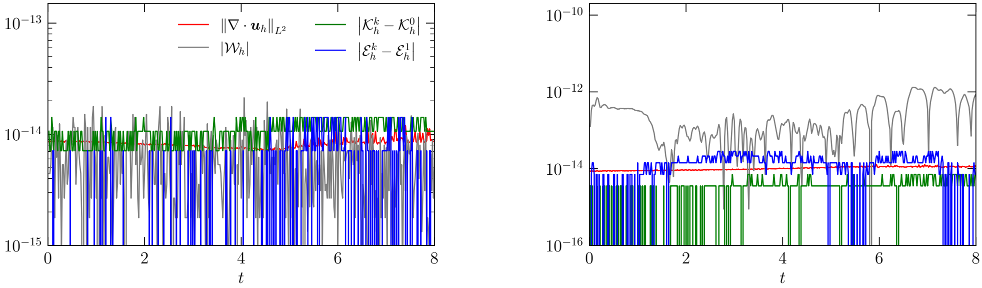

Figure – Discrete mass, energy, enstrophy and vorticity conservation over time of the ideal shear layer roll-up test for $N = 2$, $c = 0$ (left), $c = 0.25$ (right), $K = 48$ and $\Delta t = \frac{1}{50}$.

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Conservation and dissipation tests : Shear layer roll-up

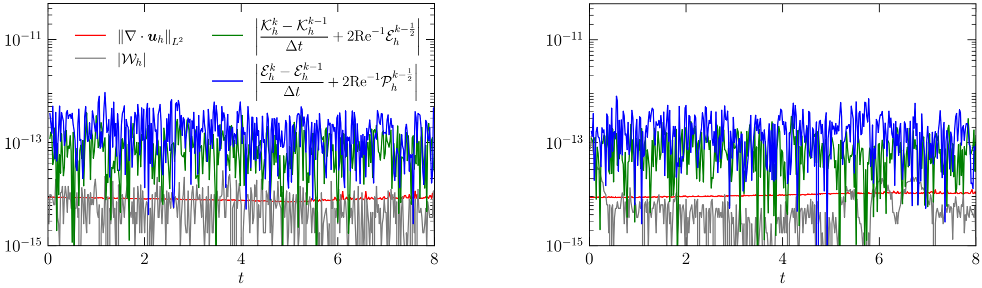

Figure – Discrete mass conservation, energy and enstrophy balances, and vorticity conservation over time of the viscous shear layer roll-up test for $N = 2$, $c = 0$ (left), $c = 0.25$ (right), $K = 48$, $\Delta t = \frac{1}{50}$ and $\mathrm{Re} = 500$.

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No-slip boundary condition test : Normal dipole collision

In $\Omega = (x, y) \in [-1, 1]^2$ with no-slip walls, the unscaled initial velocity field is given by

$$u^0 = -\frac{1}{2}\left|\omega_e\right|(y - y_1)\, e^{-(r_1/r_0)^2} + \frac{1}{2}\left|\omega_e\right|(y - y_2)\, e^{-(r_2/r_0)^2},$$
$$v^0 = -\frac{1}{2}\left|\omega_e\right|(x - x_2)\, e^{-(r_2/r_0)^2} + \frac{1}{2}\left|\omega_e\right|(x - x_1)\, e^{-(r_1/r_0)^2},$$

where $|\omega_e| = 320$, $(x_1, y_1) = (0, 0.1)$ and $(x_2, y_2) = (0, -0.1)$, r_1 and r_2 are distances to (x_1, y_1) and (x_2, y_2), respectively, and $r_0 = 0.1$.

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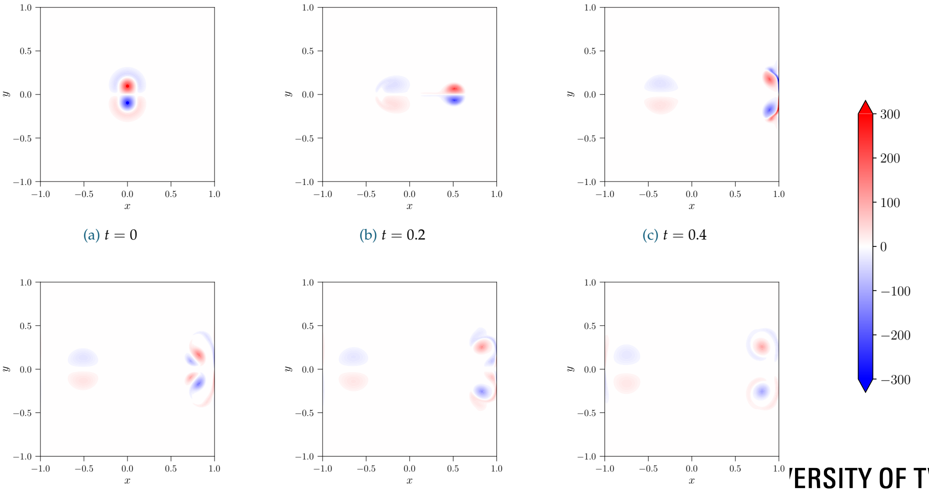
No-slip boundary condition test : Normal dipole collision



(a) $t = 0$ (b) $t = 0.2$ (c) $t = 0.4$

(d) $t = 0.6$ (e) $t = 0.8$ (f) $t = 1$

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No-slip boundary condition test : Normal dipole collision

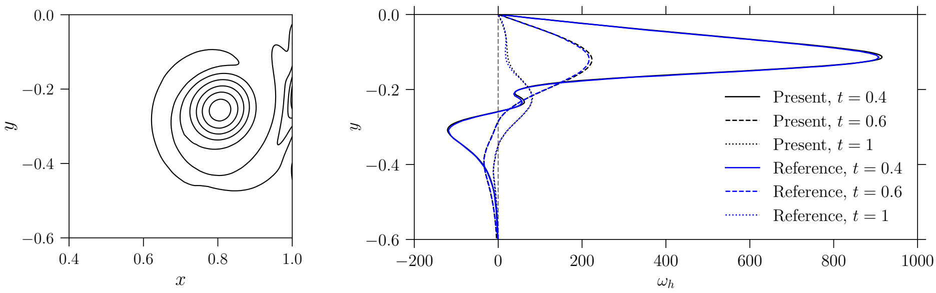

Figure – Vorticity field ω_h in region $(x, y) \in [0.4, 1] \times [-0.6, 0]$ at $t = 1$ with contour lines for $\omega_h \in \{-90, -70, -50, \cdots, 70\}$ and on the boundary section $(x, y) \in -1 \times [-0.6, 0]$ at $t \in \{0.4, 0.6, 1\}$ compared to reference results for Re $= 625$. The present simulation has 145^2 degrees of freedom for vorticity. The reference simulation uses a pseudospectral method and has 256^2 degrees of freedom for vorticity.

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No-slip boundary condition test : Normal dipole collision

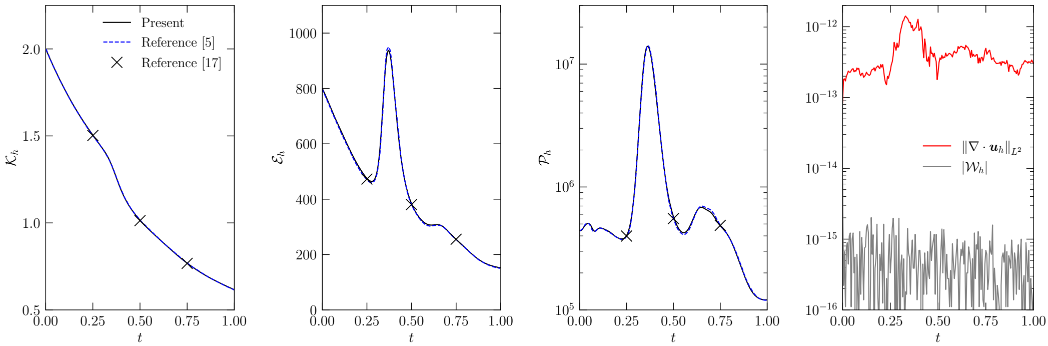

Figure – Discrete energy, enstrophy, palinstrophy over time compared to reference results and mass and vorticity conservation over time.

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Convective term, $\langle \omega_h \times \boldsymbol{u}_h, \nabla \times \omega_h \rangle_\Omega$, for enstrophy conservation

In the periodic unit square, given two random smooth scalar fields,

$$\omega = 2\pi \sin(2\pi x + e) \sin(2\pi y + f), \quad \psi = 2\pi \sin(2\pi x + g) \sin(2\pi y + h),$$

where $e, f, g, h \in (0,1)$ are random real numbers. $\boldsymbol{u}_h = \nabla \times \psi_h$.

Table – $\langle \omega_h \times \boldsymbol{u}_h, \nabla \times \omega_h \rangle_\Omega$ for $c \in \{0, 0.25\}$, $K = 12$, $N \in \{2,3\}$ and $N_Q \in \{1,2,3,4,5\}$.

N_Q \ c	0		0.25	
N	2	3	2	3
1	$1.051603e-12$	$4.718004e-12$	$-1.911286e+02$	$-6.932361e+00$
2	$-1.592504e-12$	$2.245315e-12$	$-5.186962e-12$	$7.431714e+00$
3	$2.806644e-13$	$-1.016076e-12$	$6.508571e-12$	$1.552191e-03$
4	$1.650236e-12$	$7.958079e-13$	$-7.460699e-14$	$-9.947598e-14$
5	$-5.215384e-12$	$1.875833e-12$	$1.696421e-12$	$5.684342e-13$

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Conclusions

- ✓ Single evolution equation formulation
- ✓ Straightforward incorporation of no-slip boundary conditions.
- x Nonlinear system.

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Thank you and reference

Thank you.

A. Palha, M. Gerritsma, A mass, energy, enstrophy and vorticity conserving (MEEVC) mimetic spectral element discretization for the 2D incompressible Navier-Stokes equations, Journal of Computational Physics 328 (2017) 200–220.

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