A mass-, kinetic energy- and helicity-conserving mimetic spectral element discretization for the incompressible Navier-Stokes equations
Part I: Periodic domains

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Discrete conservation properties in CFD

Motivation

A mass-, kinetic energy- and helicity-conserving high order method is of significant value in the field of CFD, in particular in the field of turbulence.

Rotation form of NSE

The *incompressible Navier-Stokes equations* (convection form) :

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1b}$$

If we use identities

$$(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2} \nabla (\boldsymbol{u} \cdot \boldsymbol{u})$$

and

$$\Delta \boldsymbol{u} = \nabla (\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}) = -\nabla \times (\nabla \times \boldsymbol{u})$$

and introduce vorticity $\boldsymbol{\omega} := \nabla \times \boldsymbol{u}$, we can get the **rotation form** :

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} + \nu \nabla \times \boldsymbol{\omega} + \nabla P = \boldsymbol{f}, \tag{2a}$$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \tag{2b}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2c}$$

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Conservation and dissipation

Kinetic energy $\mathcal{K}$, enstrophy $\mathcal{E}$, helicity $\mathcal{H}$:

$$\mathcal{K} = \frac{1}{2}\left\langle \boldsymbol{u}, \boldsymbol{u} \right\rangle_\Omega , \quad \mathcal{E} = \frac{1}{2}\left\langle \boldsymbol{\omega}, \boldsymbol{\omega} \right\rangle_\Omega , \quad \mathcal{H} = \left\langle \boldsymbol{u}, \boldsymbol{\omega} \right\rangle_\Omega .$$

Mass conservation : $\nabla \cdot \boldsymbol{u} = 0$

Given a conservative external body force $\boldsymbol{f}$,

Kinetic energy conservation and dissipation :

In inviscid case $\nu = 0$, $\dfrac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = 0$; in viscous case $\nu \neq 0$, $\dfrac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -2\nu\mathcal{E}$.

Helicity conservation and dissipation (or generation) :

In inviscid case $\nu = 0$, $\dfrac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0$; in viscous case $\nu \neq 0$, $\dfrac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -2\nu\left\langle \boldsymbol{\omega}, \nabla \times \boldsymbol{\omega} \right\rangle_\Omega$.

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Dual field : Hilbert spaces

$$\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, leftrightarrow, "\star"'] & H(\mathrm{curl};\Omega) \arrow[r, "\nabla\times"] \arrow[d, leftrightarrow, "\star"'] & H(\mathrm{div};\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, leftrightarrow, "\star"'] & L^2(\Omega) \arrow[r] \arrow[d, leftrightarrow, "\star"'] & 0 \\
0 & L^2(\Omega) \arrow[l] & H(\mathrm{div};\Omega) \arrow[l, "\nabla\cdot"'] & H(\mathrm{curl};\Omega) \arrow[l, "\nabla\times"'] & H^1(\Omega) \arrow[l, "\nabla"'] & \mathbb{R} \arrow[l, hook']
\end{tikzcd}$$

Figure – A double de Rham complex of Hilbert spaces.

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"A *mass, energy, enstrophy* and *vorticity* conserving (MEEVC) mimetic spectral element discretization for the 2D incompressible Navier-Stokes equations" [3]

A system of two evolution equations are used in MEEVC scheme :

$$\frac{\partial \boldsymbol{u}}{\partial t} + \omega \times \boldsymbol{u} + \nu \nabla \times \omega + \nabla P = 0, \tag{3a}$$

$$\frac{\partial \omega}{\partial t} + \frac{1}{2} \left(\boldsymbol{u} \cdot \nabla \right) \omega + \frac{1}{2} \nabla \cdot \left(\boldsymbol{u} \omega \right) = \Delta \omega, \tag{3b}$$

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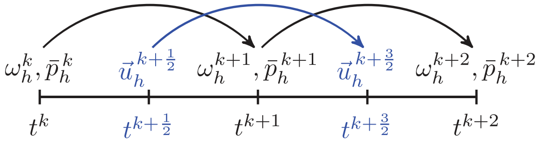

A mass-, kinetic energy- and helicity-conserving dual-field formulation

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Variational analysis will reveal that either subset of the solution, $(\boldsymbol{\omega}_1, \boldsymbol{u}_2, P_3)$ or $(P_0, \boldsymbol{u}_1, \boldsymbol{\omega}_2)$, weakly solves the rotation form of the Navier-Stokes equations ; it is a *dual-field* system.

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Mass conservation

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$$\left\langle \nabla\cdot\boldsymbol{u}_2,\epsilon_3\right\rangle_\Omega = 0,\ \forall\epsilon_3\in L^2(\Omega)$$

ensure that the relation

$$\boldsymbol{u}_2\in H(\mathrm{div};\Omega)\xrightarrow{\nabla\cdot}0\in L^2(\Omega)$$

is exact. Therefore, mass conservation is satisfied for velocity $\boldsymbol{u}_2$.

Mass conservation is only weakly satisfied for $\boldsymbol{u}_1\in H(\mathrm{curl};\Omega)$.

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Kinetic energy conservation

Given a conservative external body force, we set $\nu = 0$. Kinetic energy conservation is equivalent to

$$\frac{\mathrm{d}\mathcal{K}_1}{\mathrm{d}t} = \left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega = 0, \quad \frac{\mathrm{d}\mathcal{K}_2}{\mathrm{d}t} = \left\langle \frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{u}_2 \right\rangle_\Omega = 0.$$

If we select $\boldsymbol{\epsilon}_1$ to be $\boldsymbol{u}_1 \in H(\mathrm{curl};\Omega)$ in the first evolution equation, we get

$$\left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega + \cancel{\langle \boldsymbol{\omega}_1 \times \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle_\Omega} + \cancel{\langle \nabla P_0, \boldsymbol{u}_1 \rangle_\Omega} = \left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega = 0.$$

The second term vanishes because of the fact that the cross product of two vectors is perpendicular to both vectors, i.e.,

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{a} \rangle_\Omega = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{b} \rangle_\Omega \equiv 0. \tag{4}$$

This relation will be used repeatedly in this work. From relation

$$\langle \boldsymbol{u}_1, \nabla \epsilon_0 \rangle_\Omega = 0, \ \forall \epsilon_0 \in H^1(\Omega),$$

It follows that $\langle \nabla P_0, \boldsymbol{u}_1 \rangle_\Omega = 0$ because $P_0 \in H^1(\Omega)$.

Similarly, we can get

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$$\frac{\mathrm{d}\mathcal{K}_1}{\mathrm{d}t} = \left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega = 0, \quad \frac{\mathrm{d}\mathcal{K}_2}{\mathrm{d}t} = \left\langle \frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{u}_2 \right\rangle_\Omega = 0.$$

If we select $\boldsymbol{\epsilon}_1$ to be $\boldsymbol{u}_1 \in H(\mathrm{curl};\Omega)$ in the first evolution equation, we get

$$\left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega + \cancel{\langle \boldsymbol{\omega}_1 \times \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle_\Omega} + \cancel{\langle \nabla P_0, \boldsymbol{u}_1 \rangle_\Omega} = \left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega = 0.$$

The second term vanishes because of the fact that the cross product of two vectors is perpendicular to both vectors, i.e.,

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{a} \rangle_\Omega = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{b} \rangle_\Omega \equiv 0. \tag{4}$$

This relation will be used repeatedly in this work. From relation

$$\langle \boldsymbol{u}_1, \nabla \epsilon_0 \rangle_\Omega = 0, \ \forall \epsilon_0 \in H^1(\Omega),$$

It follows that $\langle \nabla P_0, \boldsymbol{u}_1 \rangle_\Omega = 0$ because $P_0 \in H^1(\Omega)$.

Similarly, we can get

$$\left\langle \frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{u}_2 \right\rangle_\Omega + \cancel{\langle \boldsymbol{\omega}_2 \times \boldsymbol{u}_2, \boldsymbol{u}_2 \rangle_\Omega} - \cancel{\langle P_3, \nabla \cdot \boldsymbol{u}_2 \rangle_\Omega} = \left\langle \frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{u}_2 \right\rangle_\Omega = 0$$

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$$\frac{\mathrm{d}\mathcal{H}_2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \boldsymbol{u}_2, \boldsymbol{\omega}_2 \right\rangle_\Omega = \left\langle \frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{\omega}_2 \right\rangle_\Omega + \left\langle \boldsymbol{u}_2, \frac{\partial \boldsymbol{\omega}_2}{\partial t} \right\rangle_\Omega = 0.$$

From the two evolution equations, one can derive

$$\left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{\omega}_1 \right\rangle_\Omega + \cancel{\left\langle \boldsymbol{\omega}_1 \times \boldsymbol{u}_1, \boldsymbol{\omega}_1 \right\rangle_\Omega} + \cancel{\left\langle \nabla P_0, \boldsymbol{\omega}_1 \right\rangle_\Omega} = \left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{\omega}_1 \right\rangle_\Omega = 0.$$

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These two equations conclude the proof for $\frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}t} = 0$. And one can find

$$\mathcal{H}_2 = \left\langle \boldsymbol{u}_2, \boldsymbol{\omega}_2 \right\rangle_\Omega = \left\langle \boldsymbol{u}_2, \nabla\times\boldsymbol{u}_1 \right\rangle_\Omega = \left\langle \boldsymbol{\omega}_1, \boldsymbol{u}_1 \right\rangle_\Omega = \mathcal{H}_1$$

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Kinetic energy and helicity dissipation

In viscous case, i.e. $\nu \neq 0$, if we repeat above analysis, the viscous terms remain. We get :

$$\frac{\mathrm{d}\mathcal{K}_1}{\mathrm{d}t} = \left\langle \frac{\partial \boldsymbol{u}_1}{\partial t}, \boldsymbol{u}_1 \right\rangle_\Omega = -\nu \left\langle \boldsymbol{\omega}_2, \nabla \times \boldsymbol{u}_1 \right\rangle_\Omega = -\nu \left\langle \boldsymbol{\omega}_2, \boldsymbol{\omega}_2 \right\rangle_\Omega = -2\nu\mathcal{E}_2 \leq 0,$$

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which is in agreement with the dissipation rate for the strong form.

As for helicity, we get

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Staggered temporal discretization

We use a lowest order *Gauss integrator* as the time integrator. For example, if we apply the integrator to an ordinary differential equation of the form

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = h\left(f(t), t\right)$$

at a time step, for example from time instant t^{k-1} to time instant t^k, we obtain

$$\frac{f^k - f^{k-1}}{\Delta t} = h\left(f(t^{k-1} + \frac{\Delta t}{2}), t^{k-1} + \frac{\Delta t}{2}\right),$$

where $\Delta t = t^k - t^{k-1}$, $f^k = f(t^k)$.

Additionally, we will use the following approximation, namely midpoint rule

$$f^{k-\frac{1}{2}} = f(t^{k-1} + \frac{\Delta t}{2}) := \frac{f^{k-1} + f^k}{2},$$

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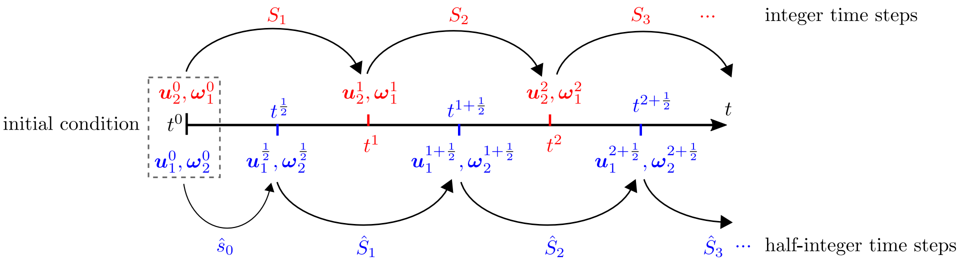

Figure – An illustration of the proposed staggered temporal scheme. The iterations proceed in a sequence : $\hat{s}_0 \rightarrow S_1 \rightarrow \hat{S}_1 \rightarrow S_2 \rightarrow \hat{S}_2 \rightarrow \cdots$.

Staggered temporal discretization

At, for example, kth integer time step S_k : Given $\left(\boldsymbol{\omega}_1^{k-1}, \boldsymbol{u}_2^{k-1}, \boldsymbol{f}^{k-\frac{1}{2}}, \boldsymbol{\omega}_2^{k-\frac{1}{2}}\right) \in H(\mathrm{curl};\Omega)\times H(\mathrm{div};\Omega)\times \left[L^2(\Omega)\right]^3 \times H(\mathrm{div};\Omega)$, find $\left(\boldsymbol{\omega}_1^{k}, \boldsymbol{u}_2^{k}, P_3^{k-\frac{1}{2}}\right) \in H(\mathrm{curl};\Omega)\times H(\mathrm{div};\Omega)\times L^2(\Omega)$ such that

$$
\begin{cases}
\left\langle \dfrac{\boldsymbol{u}_2^k - \boldsymbol{u}_2^{k-1}}{\Delta t}, \boldsymbol{\epsilon}_2 \right\rangle_\Omega + \left\langle \boldsymbol{\omega}_2^{k-\frac{1}{2}} \times \dfrac{\boldsymbol{u}_2^{k-1} + \boldsymbol{u}_2^k}{2}, \boldsymbol{\epsilon}_2 \right\rangle_\Omega & \\
\quad + \nu \left\langle \nabla \times \dfrac{\boldsymbol{\omega}_1^{k-1} + \boldsymbol{\omega}_1^k}{2}, \boldsymbol{\epsilon}_2 \right\rangle_\Omega - \left\langle P_3^{k-\frac{1}{2}}, \nabla\cdot\boldsymbol{\epsilon}_2 \right\rangle_\Omega = \left\langle \boldsymbol{f}^{k-\frac{1}{2}}, \boldsymbol{\epsilon}_2 \right\rangle_\Omega & \forall \boldsymbol{\epsilon}_2 \in H(\mathrm{div},\Omega), \\
\left\langle \boldsymbol{u}_2^k, \nabla\times\boldsymbol{\epsilon}_1 \right\rangle_\Omega - \left\langle \boldsymbol{\omega}_1^k, \boldsymbol{\epsilon}_1 \right\rangle_\Omega = 0 & \forall \boldsymbol{\epsilon}_1 \in H(\mathrm{curl};\Omega), \\
-\left\langle \nabla\cdot\boldsymbol{u}_2^k, \epsilon_3 \right\rangle_\Omega = 0 & \forall \epsilon_3 \in L^2(\Omega),
\end{cases}
$$

where $\boldsymbol{\omega}_2^{k-\frac{1}{2}}$ is "borrowed" from the other time step sequence.

Staggered temporal discretization

At, for example, kth half-integer time step $\hat{S}_k$: Given $\left(\boldsymbol{u}_1^{k-\frac{1}{2}}, \boldsymbol{\omega}_2^{k-\frac{1}{2}}, \boldsymbol{f}^k, \boldsymbol{\omega}_1^k\right) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega) \times \left[L^2(\Omega)\right]^3 \times H(\mathrm{curl};\Omega)$, seek $\left(P_0^k, \boldsymbol{u}_1^{k+\frac{1}{2}}, \boldsymbol{\omega}_2^{k+\frac{1}{2}}\right) \in H^1(\Omega) \times H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ such that

$$
\begin{cases}
\left\langle \dfrac{\boldsymbol{u}_1^{k+\frac{1}{2}} - \boldsymbol{u}_1^{k-\frac{1}{2}}}{\Delta t}, \boldsymbol{\epsilon}_1 \right\rangle_\Omega + \left\langle \boldsymbol{\omega}_1^k \times \dfrac{\boldsymbol{u}_1^{k-\frac{1}{2}} + \boldsymbol{u}_1^{k+\frac{1}{2}}}{2}, \boldsymbol{\epsilon}_1 \right\rangle_\Omega & \\
\qquad + \nu \left\langle \dfrac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2}, \nabla \times \boldsymbol{\epsilon}_1 \right\rangle_\Omega + \left\langle \nabla P_0^k, \boldsymbol{\epsilon}_1 \right\rangle_\Omega = \left\langle \boldsymbol{f}^k, \boldsymbol{\epsilon}_1 \right\rangle_\Omega & \forall \boldsymbol{\epsilon}_1 \in H(\mathrm{curl};\Omega), \\
\left\langle \nabla \times \boldsymbol{u}_1^{k+\frac{1}{2}}, \boldsymbol{\epsilon}_2 \right\rangle_\Omega - \left\langle \boldsymbol{\omega}_2^{k+\frac{1}{2}}, \boldsymbol{\epsilon}_2 \right\rangle_\Omega = 0 & \forall \boldsymbol{\epsilon}_2 \in H(\mathrm{div};\Omega), \\
\left\langle \boldsymbol{u}_1^{k+\frac{1}{2}}, \nabla \epsilon_0 \right\rangle_\Omega = 0 & \forall \epsilon_0 \in H^1(\Omega),
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Conservation properties after temporal discretization

Mass conservation is not influenced by the temporal discretization.

Givan a conservative external body force, when $\nu = 0$, one can find *kinetic energy conservation* is of the format :

$$\mathcal{K}_1^{k+\frac{1}{2}} = \frac{1}{2}\left\langle \boldsymbol{u}_1^{k+\frac{1}{2}}, \boldsymbol{u}_1^{k+\frac{1}{2}} \right\rangle_\Omega = \frac{1}{2}\left\langle \boldsymbol{u}_1^{k-\frac{1}{2}}, \boldsymbol{u}_1^{k-\frac{1}{2}} \right\rangle_\Omega = \mathcal{K}_1^{k-\frac{1}{2}}.$$

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And *helicity conservation* becomes

$$\mathcal{H}_1^k = \left\langle \frac{\boldsymbol{u}_1^{k-\frac{1}{2}} + \boldsymbol{u}_1^{k-\frac{3}{2}}}{2}, \boldsymbol{\omega}_1^{k-1} \right\rangle_\Omega = \left\langle \boldsymbol{u}_1^k, \boldsymbol{\omega}_1^k \right\rangle_\Omega = \left\langle \boldsymbol{u}_2^k, \boldsymbol{\omega}_2^k \right\rangle_\Omega = \left\langle \boldsymbol{u}_2^k, \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2} \right\rangle_\Omega = \mathcal{H}_2^k.$$

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Dissipation properties after temporal discretization

Givan a conservative external body force, when $\nu \neq 0$, by repeating aforementioned analysis, one now can find that kinetic energy dissipates at the rate

$$\frac{\mathcal{K}_1^{k+\frac{1}{2}} - \mathcal{K}_1^{k-\frac{1}{2}}}{\Delta t} = -\nu \left\langle \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2}, \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2} \right\rangle_\Omega = -2\nu\mathcal{E}_2^k \leq 0,$$

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Mimetic spatial discretization

In order to ensure the conservations at the fully discrete level, finite dimensional function spaces employed for the spatial discretization need to form a **discrete de Rham complex**,

$$\mathbb{R} \hookrightarrow H^1(\Omega^h) \xrightarrow{\nabla} H(\mathrm{curl};\Omega^h) \xrightarrow{\nabla\times} H(\mathrm{div};\Omega^h) \xrightarrow{\nabla\cdot} L^2(\Omega^h) \to 0,$$

We called these spaces *structure-preserving* or *mimetic* spaces.

✓ Once this is the case, the proofs for the conservation properties and the derivations for kinetic energy dissipation and helicity dissipation (or generation) at the semi-discrete level must hold at the fully discrete level. □

We have used the *mimetic polynomial spaces* [1, 2] as our mimetic spaces for the tests.

Mimetic spatial discretization

In order to ensure the conservations at the fully discrete level, finite dimensional function spaces employed for the spatial discretization need to form a **discrete de Rham complex**,

$$\mathbb{R} \hookrightarrow H^1(\Omega^h) \xrightarrow{\nabla} H(\mathrm{curl};\Omega^h) \xrightarrow{\nabla\times} H(\mathrm{div};\Omega^h) \xrightarrow{\nabla\cdot} L^2(\Omega^h) \to 0,$$

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Test 1 : Manufactured conservation test [4]

In periodic unit cube $\Omega := [0,1]^3$, $\boldsymbol{f} = \boldsymbol{0}$, $\nu = 0$, we use $\boldsymbol{u}|_{t=0} = \{\cos(2\pi z),\ \sin(2\pi z),\ \sin(2\pi x)\}^\mathsf{T}$ as the initial condition and let the flow evolve.

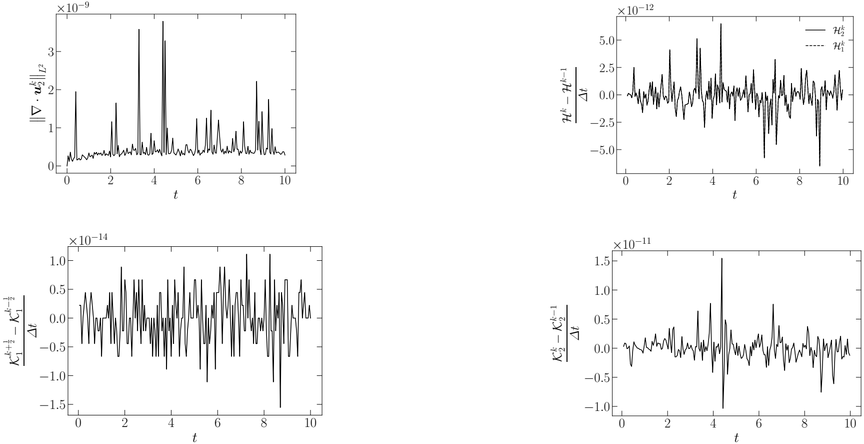


Figure – Conservation test MDF-8p2 (proposed Mimetic Dual Field method using 8^3 uniform elements of polynomial degree 2).

Test 2 : Manufactured convergence test [4]

In periodic unit cube $\Omega := [0,1]^3$, we use

$$\boldsymbol{u} = \left\{ (2-t)\cos(2\pi z),\ (1+t)\sin(2\pi z),\ (1-t)\sin(2\pi x) \right\}^{\mathsf{T}}$$

and

$$p = \sin(2\pi(x+y+t))$$

as exact solutions ($\boldsymbol{\omega}$ and $\boldsymbol{f}$ then can be calculated) and compute the flow from $t=0$ to $t=2$ and measure the error at $t=2$.

Test 2 : Manufactured convergence test [4]

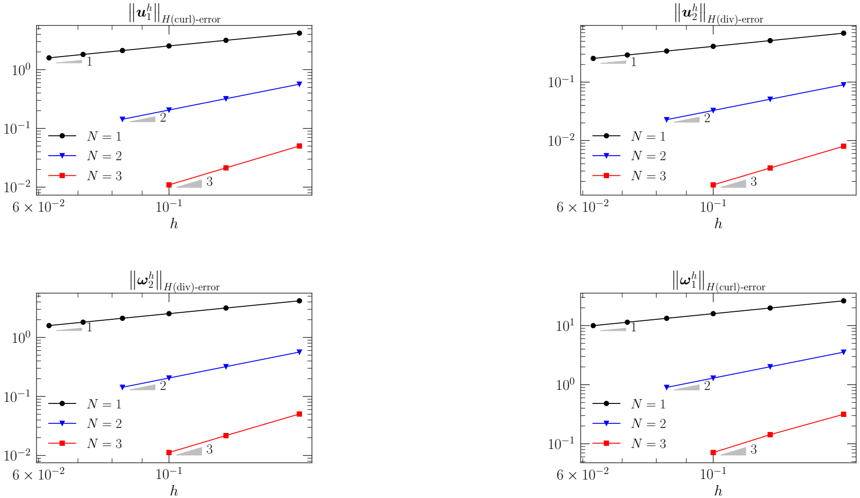


Figure – Manufactured convergence test results 1.

Test 2 : Manufactured convergence test [4]

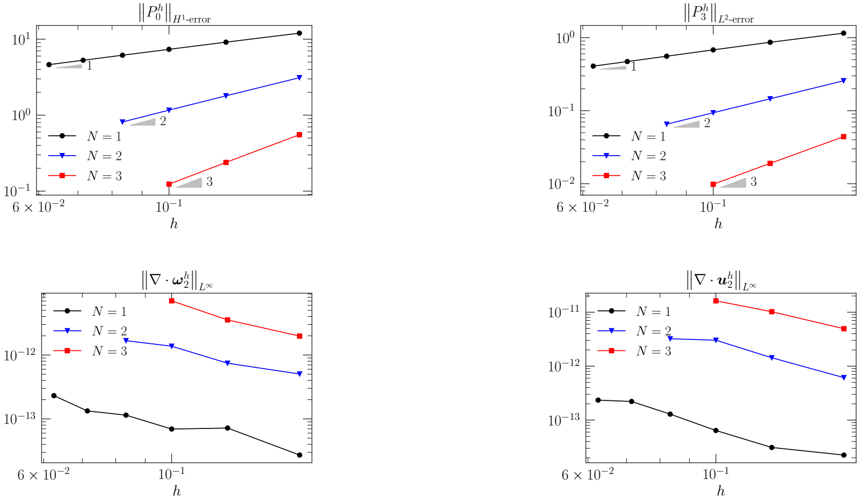


Figure – Manufactured convergence test results 2.

Test 3 : Taylor-Green vortex flow [5]

The periodic domain is given as $\Omega := [-\pi L, \pi L]^3$. The body force is set to $\boldsymbol{f} = \boldsymbol{0}$ and the initial condition is selected to be

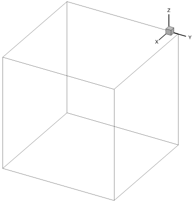
$$\boldsymbol{u}|_{t=0} = \left\{v_0 \sin(x/L)\cos(y/L)\cos(z/L), -v_0\cos(x/L)\sin(y/L)\cos(z/L), 0\right\}^{\mathsf{T}}.$$

Such an initial condition has zero initial helicity, $\mathcal{H}|_{t=0} = 0$.

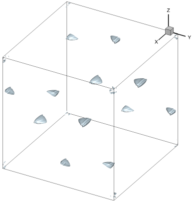
Reynolds number for this test case is defined as

$$Re = \frac{v_0 L}{\nu}.$$

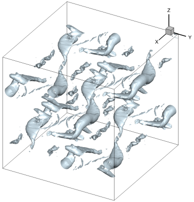
Test 3 : Taylor-Green vortex flow ; Re=500



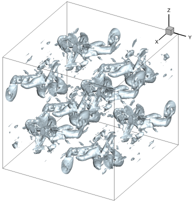
(a) $t = 0$



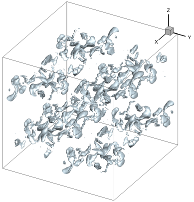
(b) $t = 3$



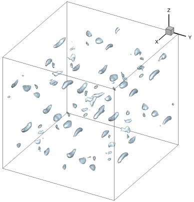
(c) $t = 6$



(d) $t = 9$



(e) $t = 12$



(f) $t = 15$

Figure – Iso-surface $\omega_1^x = -3$ of the TGV flow for MDF-24p3 at Re=500.

Test 3 : Taylor-Green vortex flow ; Re=500

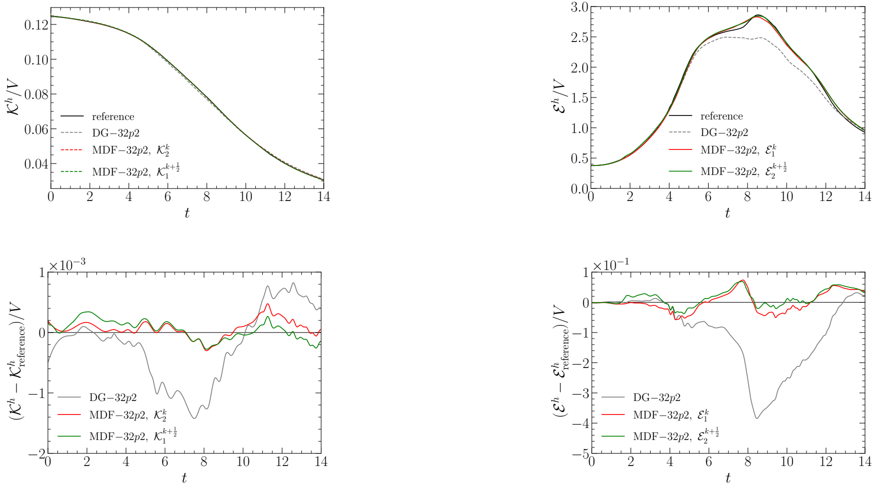


Figure – Kinetic energy and enstrophy results at Re=500.

Test 3 : Taylor-Green vortex flow ; Re=500

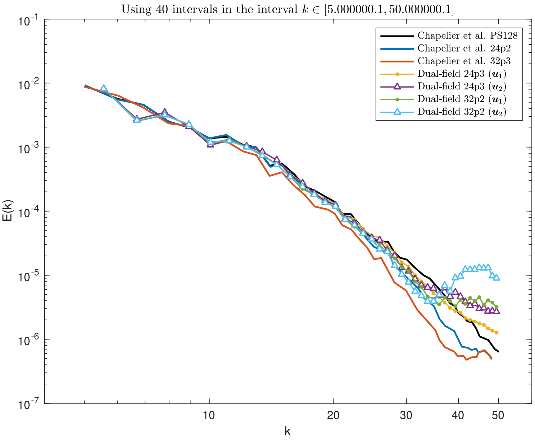


Figure – Kinetic energy spectra at t=9.1 for Re=500.

Test 3 : Taylor-Green vortex flow ; Re=1600

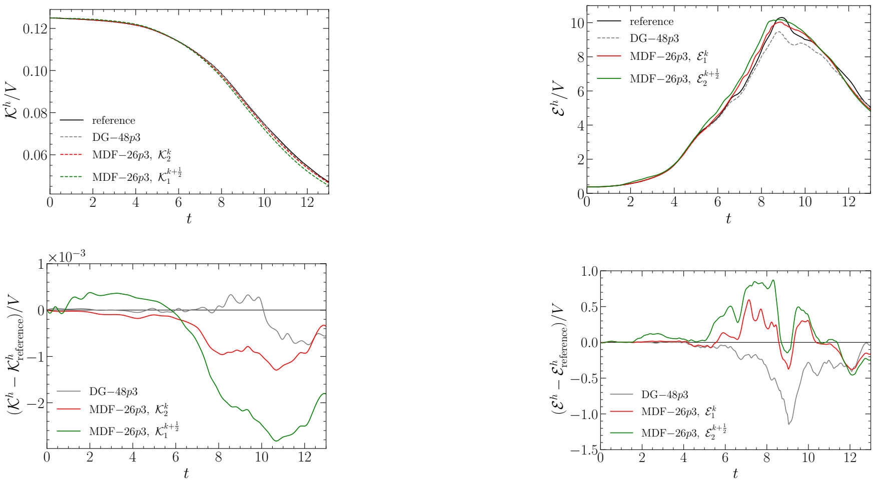


Figure – Kinetic energy and enstrophy results at Re=500.

Test 3 : Taylor-Green vortex flow ; Re=1600

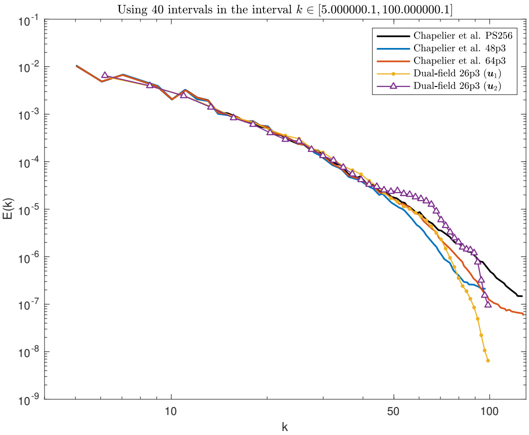


Figure – Kinetic energy spectra at t=8.2 for Re=1600.

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